6. Deformation Theory I. Deformation theory expresses properties of moduli spaces in terms of cohomology groups by studying families over local rings, especially Artinian local rings. As our first foray into this subject, we will characterize the Zariski tangent space to the Quot scheme, give a criterion for smoothness and finally a dimension estimate.

Throughout this section, X will be a nonsingular projective variety over a field k, and the Quot scheme $\mathrm{Quot}(\mathcal{O}_X^n, P(d))$ is therefore a projective scheme over k. Let q be a k-rational point of the Quot scheme and let $q : \mathcal{O}_X^n \to \mathcal{Q}$ be the corresponding quotient, with kernel $\mathcal{K}$ determined up to isomorphism.

Theorem 1: The Zariski tangent space to $\mathrm{Quot}(\mathcal{O}_X^n, P(d))$ at q is isomorphic to $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q})$ as a vector space over k.

Theorem 2: If $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q}) = 0$ then $\mathrm{Quot}(\mathcal{O}_X^n, P(d))$ is nonsingular at q.

Theorem 3: In any case,

$$\dim_q \mathrm{Quot}(\mathcal{O}_X^n, P(d)) \geq \dim_k(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q})) - \dim_k(\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q}))$$

and equality implies $\mathrm{Quot}(\mathcal{O}_X^n, P(d))$ is a local complete intersection at q.

Corresponding to the theorems are three "commutative algebra" lemmas which are valid for any k-rational point y in a k-scheme Y of finite type.

Lemma 1: The set of morphisms $f : \mathrm{Spec}(k[\epsilon]/\epsilon^2) \to Y$ extending $y \in Y$ may be naturally identified with the Zariski tangent space to Y at y.

Lemma 2: $\mathcal{O}_y$ is regular if and only if for every surjection $B \to A$ of local Artinian rings with residue field k, every element of $\mathrm{Hom}_k(\mathcal{O}_y, A)$ lifts to an element of $\mathrm{Hom}_k(\mathcal{O}_y, B)$.

Lemma 3: $\mathcal{O}_y$ is a quotient $f : R \to \mathcal{O}_y$ of a regular local k-algebra R, such that f induces an isomorphism on Zariski tangent spaces.

(In particular, the dimension of R is equal to $\dim_k(m_y/m_y^2)$.)

Proof of Theorem 1: The natural identification of Lemma 1 goes as follows. The morphisms f extending $y \in Y$ correspond to local homomorphisms $f^\# : \mathcal{O}_y \to k[\epsilon]/\epsilon^2$. These are equivalent to linear maps $m_y/m_y^2 \to \epsilon k$, hence to the Zariski tangent space. For maps to the Quot scheme, each such f is a quotient $\overline{q} : \mathcal{O}^n_{X_{k[\epsilon]/\epsilon^2}} \to \overline{\mathcal{Q}}$ extending the given quotient $q : \mathcal{O}_X^n \to \mathcal{Q}$. We thus need to characterize such quotients.

We first work locally. Suppose T is a finitely generated k-algebra and A is a Noetherian local ring with residue field k. Then a finitely generated module M over $T_A := A \otimes_k T$ is flat over A if and only if:

$$0 = \mathrm{Tor}_1^A(k, M) = \ker(m_A \otimes_A M \to M)$$

Now suppose $T_A^n \to \overline{Q}$ is a flat quotient of T_A-modules with kernel $\overline{K}$ which reduces to $T^n \to Q$ (with kernel K) modulo m_A. Then flatness of $\overline{Q}$ is equivalent to the exactness of the sequence of reductions modulo m_A: $0 \to k \otimes_A \overline{K} \to T^n \to Q \to 0$, i.e. to the condition that $\overline{K}$ reduce to K.

It is also equivalent, by the five lemma, to the condition that the left vertical map below be surjective (it is automatically injective)

$$\begin{array}{ccccccccc} 0 & \to & \ker & \to & m_A \otimes_A T_A^n & \to & m_A \otimes_A \overline{Q} & \to & 0 \\ & & f \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \overline{K} \cap m_A T_A^n & \to & m_A T_A^n & \to & m_A \overline{Q} & \to & 0 \end{array}$$

But the image of f is $m_A \overline{K}$, so this really says that:

$$m_A \overline{K} = \overline{K} \cap m_A T_A^n$$

is a criterion for flatness of $\overline{Q}$. One example of a flat $\overline{Q}$ is the trivial quotient:

$$(q \otimes 1) : T_A^n \to Q_A \quad \text{with kernel} \quad K_A = K + m_A K$$

which indeed satisfies $m_A K_A = m_A K = K_A \cap m_A T_A^n$.

Now set $A = k[\epsilon]/\epsilon^2$ and choose generators and relations for K:

$$R \xrightarrow{r} G \xrightarrow{g} T^n$$

That is, R and G are free R-modules, $\ker(g) = \mathrm{im}(r)$ and $\mathrm{im}(g) = K$. Note that if $\overline{q} : T_A^n \to \overline{Q}$ is a flat extension of q, then by Nakayama's Lemma, the generators lift to generators of $\overline{K}$ since $\overline{K}$ reduces to K modulo m_A. Likewise by flatness of $\overline{K}$, the relations lift to relations, giving a presentation of $\overline{K}$:

$$R_A \xrightarrow{\overline{r}} G_A \xrightarrow{\overline{g}} T_A^n$$

Conversely, let $\overline{r}$ and $\overline{g}$ be arbitrary lifts of r and g. Then:

Assertion 1: The map $(q \otimes 1) \circ \overline{g} \circ \overline{r} : R_A \to Q_A$ does not depend upon $\overline{r}$ and descends to a map of T-modules:

$$e(\overline{g}) : R \to \epsilon Q \text{ for } R = R_A/\epsilon R_A \text{ and } \epsilon Q \subset Q_A$$

which vanishes if and only if $\mathrm{coker}(\overline{g})$ is flat over A.

Proof: Since $\overline{g} \circ \overline{r}$ is zero modulo ϵ and $\epsilon^2 = 0$, it follows that the map descends as indicated. Since $(q \otimes 1) \circ \overline{g}$ is zero modulo ϵ, it does not depend upon the lift of $\overline{r}$. If $\mathrm{coker}(\overline{g})$ is flat, we've already seen that we can find a lift $\overline{r}$ so that $\overline{g} \circ \overline{r} = 0$. Thus $e(\overline{g}) = (q \otimes 1) \circ \overline{g} \circ \overline{r} = 0$ in that case.

For an arbitrary lift $\overline{g}$, let $\overline{K}$ be its image. Then for each $\overline{r}$, the image of $\overline{g} \circ \overline{r}$ is in $\overline{K} \cap \epsilon T_A^n$, and if $e(\overline{g}) = 0$, then it is additionally in $\epsilon K = \epsilon \overline{K}$. Thus to show that $\mathrm{coker}(\overline{g})$ is flat by the earlier criterion for flatness, we need to show that $\epsilon K = \overline{K} \cap \epsilon T_A^n$. To do this, it suffices to show that given $z \in \overline{K} \cap \epsilon T_A^n$ and a lift $\overline{r}$, then there is an $x \in R_A$ so that $\overline{g} \circ \overline{r}(x) - z \in \epsilon K$.

Choose $y \in G_A$ so that $\overline{g}(y) = z$. Then the image of y in G is in the image of r, so we can find an $x \in R_A$ with the same image in G. But then $\overline{r}(x) - y \in \epsilon G$, hence $\overline{g} \circ \overline{r}(x) - z \in \epsilon K$ as desired.

Given a lift $\overline{g}$ satisfying $e(\overline{g}) = 0$, then $(q \otimes 1) \circ \overline{g}$ descends to a map:

$$\phi(\overline{g}) : K \to \epsilon Q$$

(it descends to $G \to \epsilon Q$ as above, and further to $\phi(\overline{g})$ because $e(\overline{g}) = 0$). Notice that this map only depends upon the image of $\overline{g}$, i.e. only upon the equivalence class of flat quotients determined by $\overline{g}$. Moreover, if $\phi(\overline{g}) = \phi(\overline{g}')$, then $\mathrm{im}(\overline{g} - \overline{g}') \subset \epsilon K \subset \overline{K} \cap \overline{K}'$, so the images are the same and thus $\phi(\overline{g})$ determines the image of $\overline{g}$. Finally, any $\phi : K \to \epsilon Q$ gives rise to a map $G_A \to Q_A$ which can be factored through a $\overline{g}$ satisfying $\phi(\overline{g}) = \phi$. Thus the $\overline{q} : T^n \to \overline{Q}$ have been identified with $\mathrm{Hom}_T(K, \epsilon Q)$.

It is easy to globalize this to X. Choose an open cover of X by affines $U_i = \mathrm{Spec}(T_i)$ over which the quotient $q : \mathcal{O}_X^n \to \mathcal{Q}$ is of the form $T_i^n \to Q_i$. Then an extension of q to a flat quotient $\overline{q} : \mathcal{O}_{X_A}^n \to \overline{\mathcal{Q}}$ gives local extensions of the quotients $T_i^n \to Q_i$ each of which corresponds to a $\phi_i \in \mathrm{Hom}_{T_i}(K_i, Q_i)$ via the local version above. The ϕ_i then patch together to a global element of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q})$ because they must agree on the overlaps. And conversely, an element of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q})$ gives local quotients $T_{iA}^n \to \overline{Q}_i$ which patch to a global extension of q. This proves Theorem 1.

Proof of Theorem 2: Translating Lemma 2 into flat quotients, we need to show that any extension of q to a quotient $\overline{q}^A : \mathcal{O}^n_{X_A} \to \overline{\mathcal{Q}}^A$ which is flat over A further extends to a quotient $\overline{q}^B : \mathcal{O}^n_{X_B} \to \overline{\mathcal{Q}}^B$ which is flat over B. It suffices to prove this under the assumption that $m_B J = 0$ where:

$$0 \to J \to B \to A \to 0$$

(which is the case, for instance, when $B = k[\epsilon]/\epsilon^2$ and $A = k$)

Again, we will start locally. If a flat quotient $\overline{q}^A : T^n_A \to \overline{Q}^A$ is given. Then $\overline{q}^B : T^n_B \to \overline{Q}^B$ reducing to $\overline{q}^A$ modulo J is flat if and only if:

$$0 = \mathrm{Tor}_1^B(A, \overline{Q}^B) = \ker(J \otimes_B \overline{Q}^B \to \overline{Q}^B)$$

As before, this is equivalent to $\overline{K}^B$ reducing to $\overline{K}^A$ modulo J, and to:

$$J \otimes_k K = J\overline{K}^B = \overline{K}^B \cap JT^n_B$$

Also as before, we can start with a presentation of $\overline{K}^A$: $R_A \xrightarrow{\overline{r}^A} G_A \xrightarrow{\overline{g}^A} T^n_A$ and attempt to further lift to a presentation with flat coker$(\overline{g})$.

Assertion 1: Suppose $R_B \xrightarrow{\overline{r}} G_B \xrightarrow{\overline{g}} T^n_B$ are arbitrary lifts of $\overline{r}^A$ and $\overline{g}^A$. Then $\overline{g} \circ \overline{r} : R_B \to T^n_B$ descends to a map from R to $J \otimes_k T^n$ and composing further with q, we obtain a map:

$$e(\overline{g}) : R \to J \otimes_k Q$$

which does not depend upon $\overline{r}$ and vanishes iff coker$(\overline{g})$ is flat.

Proof: This basically is the same as the previous Assertion 1, except that the map $J \otimes_k T^n \to J \otimes_k Q$ does not descend from a trivial quotient extending q because the trivial quotient probably does not extend $\overline{q}^A$(!) Absent this, a couple of the arguments need to be slightly modified. If $\overline{r}$ and $\overline{r}'$ are two lifts of $\overline{r}^A$, then the image of $\overline{r} - \overline{r}'$ lies in $J \otimes_k G$, and the further image under $\overline{g}$ lies in $J \otimes_k K$, showing that $e(\overline{g})$ does not depend upon $\overline{r}$. Also, if $\overline{g} \circ \overline{r} = 0$, then it is clear that $e(\overline{g}) = 0$. The rest of the argument is the same.

Corollary: If the set of quotients $\overline{q}$ extending $\overline{q}^A$ is nonempty, then it is an affine space with underlying vector space $\mathrm{Hom}_T(K, Q) \otimes_k J$.

Proof: If a single quotient exists, denote it by $\overline{q}^B$, and use it to define:

$$\phi(\overline{g}) : K \to J \otimes_k Q$$

as before to be the descendant of $\overline{q}^B \circ \overline{g}$. Then as before, we get an identification of $\mathrm{Hom}_T(K, Q) \otimes_k J$ with the set of equivalence classes of flat quotients extending $\overline{q}^A$, and $\overline{q}^B$ corresponds to the zero element of this vector space. But the choice of $\overline{q}^B$ was not canonical (unlike the situation of Theorem 1) so we only get an affine space from this identification.

Next, we need to investigate the "obstruction" to the existence of a $\overline{q}^B$.

Assertion 2: Given $\overline{g}$, then the cokernel: $R \overset{r+e(\overline{g})}{\to} G + J \otimes_k Q \to E(\overline{g}) \to 0$ fits into an exact sequence:

$$0 \to J \otimes_k Q \to E(\overline{g}) \to K \to 0$$

and the corresponding extension class:

$$\epsilon(\overline{q}^A) \in \mathrm{Ext}^1(K, Q \otimes_k J)$$

only depends upon $\overline{q}^A$ and only vanishes if a flat $\overline{q}^B$ extending $\overline{q}^A$ exists.

Proof: Lots of homological algebra.

Now for the global version. Let $U_i = \mathrm{Spec}(T_i)$ be an open affine cover of X with intersections $U_i \cap U_j = \mathrm{Spec}(T_{ij})$. A global quotient $\overline{q}^A : \mathcal{O}^n_{X_A} \to \overline{\mathcal{Q}}^A$ gives rise to local quotients $\overline{q}^A_i : T^n_{i\ A} \to \overline{Q}^A_i$ hence to local extension classes $\epsilon(\overline{q}^A_i) \in \mathrm{Ext}^1_{T_i}(K_i, Q_i) \otimes_k J$ which agree on the overlaps, patching to a global extension class $\epsilon(\overline{q}^A) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q}) \otimes_k J$. From the local-to-global spectral sequence for Ext, it follows that there are maps:

$$\mathrm{H}^1(X, Hom(\mathcal{K}, \mathcal{Q})) \otimes_k J \hookrightarrow \mathrm{Ext}^1(\mathcal{K}, \mathcal{Q}) \otimes_k J \to \mathrm{H}^0(X, Ext^1(\mathcal{K}, \mathcal{Q})) \otimes_k J \to \ldots$$

If $\epsilon(\overline{q}^A)$ has zero image in $\mathrm{H}^0(X, Ext^1(\mathcal{K}, \mathcal{Q})) \otimes_k J$, then the local $\epsilon(\overline{q}^A_i)$ are zero, hence there are local lifts $\overline{g}_i$ giving rise to flat quotients $\overline{q}^B_i$. These local quotients may not patch to a global quotient, but their differences over $\mathrm{Spec}(T_{ij})$ may be interpreted as elements $\phi_{ij} \in \mathrm{Hom}_{T_{ij}}(K_{ij}, Q_{ij})$ which vanish if and only $\overline{q}^B_i$ patches with $\overline{q}^B_j$ by the corollary to Assertion 1. It is not hard to see that the ϕ_{ij} are a cocycle, i.e.

$$\{\phi_{ij}\} \in \mathrm{H}^1(X, Hom(\mathcal{K}, \mathcal{Q})) \otimes J$$

whose image in the $\mathrm{Ext}^1(\mathcal{K},\mathcal{Q})$ is $\epsilon(\overline{q}^A)$. Thus if this global obstruction class is zero, then the cocycle is a coboundary, satisfying $\phi_{ij} = \phi_i - \phi_j$ for $\phi_i \in \mathrm{Hom}(K_i, Q_i) \otimes_k J$. We can use the ϕ_i to modify our $\overline{g}_i$ (again using the corollary to Assertion 1) to produce local quotients that do patch to a global quotient. Thus Theorem 2 is proved!

Remark: It is perfectly possible to have a proper subspace $V \subset \mathrm{Ext}^1(\mathcal{K},\mathcal{Q})$ with the property that every $\epsilon(\overline{\mathcal{Q}}^A) \in V \otimes J$. We define:

$$\mathrm{Obst}(q) \subset \ \mathrm{Ext}^1(\mathcal{K},\mathcal{Q})$$

to be the smallest such subspace, called the obstruction space, and note that the previous proof shows that if the obstruction space is zero, then $\mathrm{Quot}(\mathcal{O}_X^n, P(d))$ is nonsingular at q. The converse is also easily seen to be true....if the Quot scheme is nonsigular, then global quotients lift, so local quotients lift, so the local obstruction classes are zero, and the resulting cocycles are zero because the local liftings patch to a global lift.

Proof of Theorem 3: Let $R \to \mathcal{O}_q$ be the surjective map of local rings from Lemma 3 and let I be the kernel. Then consider the exact sequence:

$$(*)\ 0 \to J \to B \to \mathcal{O}_q \to 0$$

where $B = R/m_R I$ and $J = I/m_R I$. Note that $Jm_B = 0$ and J is a finite dimensional vector space over k of dimension equal to the number of generators of I. A "universal" quotient $\overline{q} : \mathcal{O}^n_{X_{\mathcal{O}_q}} \to \overline{Q}$ over $\mathcal{O}_q$ exists, so we are in a position to use the machinery of the proof of Theorem 2. If we prove that $\dim_k(J) \leq \dim_k(\mathrm{Ext}^1(\mathcal{K},\mathcal{Q}))$ then we will have proved Theorem 3. In fact, we will prove that $\dim_k(J) \leq \dim_k(\mathrm{Obst}(q))$ which is somewhat better.

Here's the idea. Elements j^* of the dual space J^* push forward $(*)$ to:

$$0 \to k \to B' \to \mathcal{O}_q \to 0$$

giving rise to obstruction classes: $\epsilon_{j^*}(\overline{q}) \in \mathrm{Obst}(q)$. Such a class cannot be zero because if it were, then there would be a flat quotient over B' which, by the universal property of the Quot scheme, would yields a map $\mathcal{O}_q \to B'$ splitting the sequence above. But this would in turn imply that the Zariski tangent space of B' (hence of B, hence of R) is larger than that of $\mathcal{O}_q$, which contradicts our choice of R. Thus we have an injective map $J^* \to \mathrm{Obst}(q)$. It only remains to show that this map is k-linear to complete the proof.

Example (Quot Schemes on Smooth Curves) Let C be a nonsingular projective curve, and consider the Quot scheme: $\mathrm{Quot}(\mathcal{O}_C^n, P(d))$. Then $P(d)$ is either linear or constant, and we will treat the two cases separately.

(a) $P(d) = a$. Each quotient $\mathcal{O}_C^n \to \mathcal{Q}_Z$ is supported on a zero-dimensional subscheme $Z \subset C$ and each kernel $\mathcal{K}$ is locally free of rank n. Thus the Zariski tangent space to $\mathrm{Quot}(\mathcal{O}_C^n, e)$ at q is:

$$\mathrm{Hom}(\mathcal{K}, \mathcal{Q}_Z) = \mathrm{H}^0(X, \mathcal{K}^* \otimes \mathcal{Q}_Z) \text{ which has constant dimension } na$$

and the Quot scheme is nonsingular since $\mathrm{Ext}^1(\mathcal{K}, \mathcal{Q}) = \mathrm{H}^1(X, \mathcal{K}^* \otimes \mathcal{Q}_Z) = 0$

The Hilbert scheme is the familiar symmetric product:

$$\mathrm{Hilb}_{C,a} = \mathrm{Sym}^a(C)$$

and in general, there is a morphism $\mathrm{Quot}(\mathcal{O}_C^n, a) \to \mathrm{Sym}^a(C)$ defined as follows. The kernel $\mathcal{K} \to \mathcal{O}^n_{C \times \mathrm{Quot}}$ of the universal quotient over the Quot scheme is locally free (this is a consequence of flatness) and of rank n. One can check that the cokernel of the top exterior power $\wedge^n \mathcal{K} \to \mathcal{O}_{C \times \mathrm{Quot}}$ is a flat family of quotients of length a. This defines the morphism.

(b) $P(d) = (n-m)d + b$ where $0 < m < n$.

Then as before, $\mathcal{K}$ is locally free but this time $\mathcal{Q}$ is not supported on a zero-dimensional subscheme when q is a point of $\mathrm{Quot}(\mathcal{O}_C^n, (n-m)d + b)$. We have already discussed the fact that we should regard this scheme as a compactification of the space of maps of a fixed degree from C to $G(m, n)$. This space can be singular! However, using the surjection:

$$\mathrm{H}^1(C, \mathcal{K}^* \otimes \mathcal{O}_C^n) \to \mathrm{H}^1(C, \mathcal{K} \otimes \mathcal{Q}) = \mathrm{Ext}^1(\mathcal{K}, \mathcal{Q})$$

we see that if $\mathrm{H}^1(C, \mathcal{K}^*) = 0$, then the Quot scheme is nonsingular at q. This is always the case if $C = \mathbf{P}^1$ (every summand of $\mathcal{K}$ has non-positive degree) but may fail if $C \neq \mathbf{P}^1$.

Example: (Points on a Smooth Surface)

Let S be a nonsingular projective surface, and consider $H_{S,a}$ the Hilbert scheme of subschemes of length a of S. If $h \in H_{S,a}$ corresponds to a quotient:

$$0 \to \mathcal{I}_Z \to \mathcal{O}_S \to \mathcal{O}_Z \to 0$$

then it is relatively easy to see that:

$$\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{I}_Z, \mathcal{O}_Z) \text{ has dimension } 2a \text{ and } \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{I}_Z, \mathcal{O}_Z) \text{ has dimension } a$$

independent of $\mathcal{O}_Z$. On the other hand, as in the curve case, $(S^a - \Delta)/\Sigma_a$ (the complement of the diagonal quotiented by the symmetric group) is a nonsingular open subset of $H_{S,a}$ of dimension $2a$ corresponding to reduced subschemes. Unlike the curve case, this does not extend to an isomorphism from $\mathrm{Sym}^a(S)$. But this does prove, together with the dimension of the Zariski tangent space computed above, that the closure of this open subset is a nonsingular connected component of the Hilbert scheme. One can additionally prove that the Hilbert scheme is connected. Notice that this is an example where the obstruction space is zero but Ext^1 is not zero!

Example: (Local Complete Intersections)

If $Z \subset X$ is a local complete interesection, then:

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z) = \mathrm{H}^1(Z, N_{Z/X})$$

where $N_{Z/X}$ is the normal bundle to Z in X. We can apply this to help us to analyze Hilbert schemes at such points. For example.

(a) If $C \subset \mathbf{P}^n$ is is a nonsingular curve of degree $e > 2g - 2$, then it is a nonsingular point of the Hilbert scheme $H_{\mathbf{P}^n, ed+1-g}$. That is because the Euler sequence gives rise to a surjection:

$$\mathrm{H}^1(C, \mathcal{O}_C^{n+1}(1)) \to \ \mathrm{H}^1(C, N_{C/\mathbf{P}^n})$$

and the former space is zero by Riemann-Roch.

(b) Pairs of skew lines in $\mathbf{P}^3$ are nonsingular points of $H_{\mathbf{P}^3, 2d+1}$ giving a component of dimension $2\dim(G(2,4)) = 8$. On the other hand, a plane conic union a point is a nonsingular point of the same Hilbert scheme, giving a component, this time of dimension $5+3+3 = 11$ (dimensions of conics in $\mathbf{P}^2$, planes in $\mathbf{P}^3$ and points in $\mathbf{P}^3$ respectively). These two components meet along the locus of degenerate conics in $\mathbf{P}^2$ with a nilpotent in the singular locus. This gives an example of a Hilbert scheme with two components of different degrees.

(c) Projected canonical curves of genus 5. Let $C \subset \mathbf{P}^3 \subset \mathbf{P}^4$ be a nonsingular canonical genus 5 curve, projected from $\mathbf{P}^4$ and reembedded in $\mathbf{P}^4$. This is a point in the Hilbert scheme $H_{\mathbf{P}^4,10d-4}$. There are two components of this Hilbert scheme meeting along such points. One of them is the component (of dimension 36) consisting of embedded canonical curves (the general one of which is not projected) and the other (also of dimension 36) is the component consisting of curves of degree 10 and genus 5 which span a $\mathbf{P}^3$ in $\mathbf{P}^4$ (the general one of which is not canonical). This gives an example of a singular point of a Hilbert scheme which parametrizes a nonsingular embedded curve.