8. Hilbert Schemes of Points on Curves: We survey a few of the known properties of Hilbert schemes of points on curves.

If X is a projective scheme over $\mathbf{C}$, let $X^{[d]}$ denote the Hilbert scheme of subschemes of X of constant Hilbert polynomial d (i.e. of length d).

When C is a compact Riemann surface then $C^{[d]}$ is the symmetric product, a nonsingular complex projective variety of dimension d.

Theorem (Macdonald): The Betti numbers $b_i(C^{[d]}) = \dim \mathrm{H}^i(C^{[d]}, \mathbf{Q})$ are computed by the following generating function:

$$\sum_{d\geq 0}\sum_{i\geq 0} b_i(C^{[d]})t^i q^d = \frac{(1+tq)^{2g}}{(1-q)(1-t^2q)}$$

Given a coherent sheaf on $C^{[d]}$, there is a natural cycle map:

$$A^*(C^{[d]}) \to \mathrm{H}^{2*}(C^{[d]}, \mathbf{C})$$

i.e. Chern classes on $C^{[d]}$ determine cohomology classes. We may use the universal family to construct several of these.

Recall that the universal family is a subscheme:

$$\mathcal{D} \subset C^{[d]} \times C$$

which is flat over $C^{[d]}$ with fibers equal to the subschemes of C of length d. Let p and q be the projection maps to $C^{[d]}$ and C respectively.

It follows from cohomology and base change that $p_*q^*\mathcal{O}_C$ is locally free on $C^{[d]}$ of rank d. The **half-diagonal class** $\frac{1}{2}\delta$ is defined by:

$$\frac{1}{2}\delta := -c_1(p_*q^*\mathcal{O}_C)$$

One can show that $\delta \in A^1(C^{[d]})$ is represented by the "big diagonal"...the closed subvariety parametrizing schemes that contain a point of multiplicity two or more. This Chern class is present on any Hilbert scheme of points.

Choose a closed point $p_0 \in C$. Then there are induced closed immersions:

$$p_0 \in C \hookrightarrow C^{[2]} \hookrightarrow C^{[3]} \hookrightarrow \ldots$$

given by "addition of the point p_0." More accurately, when we take the quotient of the inclusion of products:

$$C^{d-1} \times \{p_0\} \subset C^d$$

by the action of the symmetric group, we get the desired closed immersion of symmetric products. We define:

$$x_{p_0} := [C^{[d-1]}] \in A^1(C^{[d]})$$

and note that

$$[C^{[d]}] \cap x_{p_0}^e = [C^{[d-e]}]$$

The classes $[C^{[d-1]}] \in A^1(C^{[d]})$ depend upon p_0 (when C has positive genus) but their images in $\mathrm{H}^2(C^{[d]}, \mathbf{C})$ do not. We will let x denote the image.

Finally, we may use the fact that $\mathcal{D} \subset C^{[d]} \times C$ is a Cartier divisor to define the **determinantal class** on $C^{[d]}$. We start first with the case $d = g-1$. We define:

$$\theta := -c_1(p_! \mathcal{O}_{C^{[g-1]} \times C}(\mathcal{D})) = c_1(R^1 p_* \mathcal{O}(\mathcal{D})) - c_1(p_* \mathcal{O}(\mathcal{D}))$$

If we choose a "base divisor" $D_0 = p_1 + ... + p_g$ consisting of g distinct points, then the sequence:

$$0 \to \mathcal{O}(\mathcal{D}) \to \mathcal{O}(\mathcal{D} + q^* D_0) \to \mathcal{O}_{q^{-1} D_0}(\mathcal{D} + q^* D_0) \to 0$$

pushes forward to a resolution:

$$0 \to p_* \mathcal{O}(\mathcal{D}) \to E \to F \to R^1 p_* \mathcal{O}(\mathcal{D}) \to 0$$

where E and F are vector bundles of rank g, and then:

$$\theta = c_1(F) - c_1(E)$$

We get well-defined cohomology classes $\theta \in \mathrm{H}^2(C^{[d]}, \mathbf{C})$ for all d by letting:

$$\theta := c_1(p_! \mathcal{O}(\mathcal{D} + q^* D_0))$$

for any divisor D_0 of degree $g - 1 - d$.

These three divisor classes on $C^{[d]}$ have **very** different properties.

(i) Each $x_{p_0} \in A^1(C^{[d]})$ is ample.

Proof: x_{p_0} pulls back to the ample divisor $\sum \pi_i^* p_0$ on the product C^d (where $\pi_i : C^d \to C$ is the projection). The cohomological criterion for ampleness then implies x_{p_0} is ample, because $C^d \to C^{[d]}$ is a finite map.

(ii) $\mathrm{H}^0(C^{[d]}, \mathcal{O}_{C^{[d]}}(\frac{n}{2}\delta)) = 0$ for odd n and is one-dimensional for even n.

(iii) θ is pulled back from an ample divisor on the Picard group. More precisely, each $\mathrm{Pic}^d(C)$ is an abelian variety of dimension g parametrizing the isomorphism classes of line bundles on C of degree d. There are Abel-Jacobi maps $C^{[d]} \to \mathrm{Pic}^d(C)$ for each d sending D to $\mathcal{O}_C(D)$. The fibers of the Abel-Jacobi maps are the linear series $|D|$. The Abel-Jacobi maps are surjective when $d \geq g$ by Riemann-Roch. Tensoring by $\mathcal{O}_C(p_0)$ gives isomorphisms $\mathrm{Pic}^d(C) \stackrel{\sim}{\to} \mathrm{Pic}^{d+1}(C)$ and a commuting diagram:

$$\begin{array}{ccccccccccc}
C & \hookrightarrow & C^{[2]} & \hookrightarrow & \cdots & \hookrightarrow & C^{[g]} & \hookrightarrow & \cdots & \hookrightarrow & C^{[2g-1]} \\
\downarrow & & \downarrow & & & & \downarrow & & & & \downarrow \\
\mathrm{Pic}^1(C) & \stackrel{\sim}{\to} & \mathrm{Pic}^2(C) & \stackrel{\sim}{\to} & \cdots & \stackrel{\sim}{\to} & \mathrm{Pic}^g(C) & \stackrel{\sim}{\to} & \cdots & \stackrel{\sim}{\to} & \mathrm{Pic}^{2g-1}(C)
\end{array}$$

(degree $2g-1$ is significant because at that point and thereafter, the Abel-Jacobi map becomes a projective bundle)

On $\mathrm{Pic}^d(C) \times C$ there is a universal line bundle $\mathcal{L}$ with the property that when restricted to a fiber of the projection map p to $\mathrm{Pic}^d(C)$, $\mathcal{L}$ is isomorphic to the line bundle parametrized by the point of $\mathrm{Pic}^d(C)$. On $\mathrm{Pic}^{g-1}(C)$, we can define a $\theta \in A^1$ determinantally as earlier. Namely, we let:

$$\theta = -c_1(p_! \mathcal{L})$$

In this case, however, $p_* \mathcal{L} = 0$ and $R^1 p_* \mathcal{L}$ is supported (and generically of rank one along) the image of $C^{[g-1]}$. It follows that on $\mathrm{Pic}^{g-1}(C)$, θ is the push-forward of $[C^{[g-1]}]$ under the Abel-Jacobi map, and on $C^{[g-1]}$, θ is the first Chern class of the the normal sheaf to the Abel-Jacobi map. One can prove that θ (defined as before but on $\mathrm{Pic}^d(C)$) is ample on each $\mathrm{Pic}^d(C)$. Thus although θ is not ample on $C^{[d]}$ whenever the Abel-Jacobi map has positive-dimensional fibers, it is the case that a high multiple of θ is base-point free with an explicit image in projective space.

Theorem: The image of x and θ in $\mathrm{H}^2(C^{[d]}, \mathbf{Q})$ generate all the Chern classes of $C^{[d]}$, if C is "of general modulus."

Example:

$$\frac{1}{2}\delta = (d+g-1)x - \theta$$

One can similarly obtain formulas for the "smaller" diagonals ([ACGH]).

Another Example: If C is of genus g and general modulus, $d \leq 2g-2$, and r is a non-negative integer with the property that $(r+1)(g-d+r) \leq g$ then in particular, $r(g-d+r) \leq d$ and the "Brill-Noether" locus:

$$C_d^r := \{D \in C^{[d]} \mid \dim(|D|) \geq r\}$$

is nonempty and has codimension $r(d-g+r)$ in $C^{[d]}$.

(This is a deep result in Brill-Noether theory...see [ACGH])

The class $[C_d^r] \in H^{2r(g-d+r)}(C^{[d]}, \mathbf{Z})$ has been computed. It is:

$$[C_d^r] = \left(\prod_{i=0}^{r} \frac{i!}{(g-d+r+i-1)!}\right) \sum_{\alpha=0}^{r} (-1)^\alpha \frac{(g-d+r+\alpha-1)!}{\alpha!(r-\alpha)!} x^\alpha \theta^{r(g-d+r)-\alpha}$$

Finally, to compute numbers, one needs to know the intersection numbers of powers of x and of θ. These have also been computed. They are:

$$\int_{C^{[d]}} x^d = 1$$

$$\int_{C^{[d]}} \theta^d = \begin{cases} 0 \text{ if } d > g \\ \frac{g!}{(g-d)!} \text{ if } d \leq g \end{cases}$$

From this, for example, one can check that $[C_d^r]$ is not the zero class(!) by intersecting it with the appropriate power of x.