2. Coherent Sheaves and Cohomology. Coherent sheaf cohomology encodes global properties of schemes. We will discuss some of the main results, particularly Serre's Theorems A and B. Serre's FAC probably remains the best reference, though it was written before the Grothendieck "revolution."

Coherent Sheaves: Quasi-coherent sheaves over a scheme "globalize" modules over a commutative ring. Locally free sheaves are the analogues of vector bundles, and invertible sheaves are the analogues of line bundles.

A sheaf $\mathcal{F}$ of abelian groups on a scheme X is a **sheaf of $\mathcal{O}_X$-modules** if each $\mathcal{F}(U)$ is a module over $\mathcal{O}_X(U)$ in such a way that the module structures are compatible with the restriction maps.

Example. If M is a module over A, then there is a natural associated sheaf of $\mathcal{O}_{\mathrm{Spec}(A)}$-modules, denoted by $\widetilde{M}$, such that:

$\widetilde{M}(U_f) \cong M_f$ on basic open sets and

$\widetilde{M}_x \cong M_{\mathcal{P}_x}$ for each $x \in \mathrm{Spec}(A)$.

A sheaf of $\mathcal{O}_X$-modules on an arbitrary scheme is **quasi-coherent** if it is locally of the form $\widetilde{M}$ on open subsets of the form $\mathrm{Spec}(A)$. It is **coherent** if, moreover, the modules M are each finitely generated A-modules. It is **locally free** if the stalks are each free $\mathcal{O}_{X,x}$-modules.

The assignment $M \mapsto \widetilde{M}$ with the natural map on morphisms is an equivalence of categories between A-modules and quasi-coherent sheaves over $\mathrm{Spec}(A)$. If A is Noetherian, it is an equivalence between finitely generated A-modules and coherent sheaves over $\mathrm{Spec}(A)$. Over an arbitrary scheme X, the category of quasi-coherent sheaves is closed under kernels, cokernels and images, and an extension of quasi-coherent sheaves is again quasi-coherent. The same is true for coherent sheaves if X is **Noetherian**, meaning that X is covered by finitely many Noetherian open affine subschemes.

If $i : X \to Y$ is a closed immersion, the **ideal sheaf** $\mathcal{I}_{X/Y}$ is the kernel:

$$0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \stackrel{i^\#}{\to} i_*\mathcal{O}_X \to 0$$

and is locally of the form $\widetilde{I}$, where $I \subset A$ are the ideals corresponding to the closed subschemes $\mathrm{Spec}(A) \cap X$ of each open $\mathrm{Spec}(A) \subset Y$. This is therefore quasi-coherent, and coherent if A is Noetherian.

Conversely, if $\mathcal{I}$ is a quasi-coherent sub-sheaf of $\mathcal{O}_Y$, then there is a uniquely determined closed subscheme $X \subset Y$ such that $\mathcal{I} = \mathcal{I}_{X/Y}$.

If $f : X \to Y$ is a morphism of schemes and $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then $f_*\mathcal{F}$ is a sheaf of $\mathcal{O}_Y$-modules via the map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$. In the special case where $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, then $f_*(\widetilde{M}) = \widetilde{M}$, where the M on the right is viewed as a B-module via the associated ring homomorphism $\phi : B \to A$. It follows that the push-forward of a quasi-coherent sheaf is again quasi-coherent.

Given a sheaf $\mathcal{G}$ on Y, recall that the inverse image sheaf $f^{-1}\mathcal{G}$ is the sheaf on X associated to the presheaf $U \mapsto \lim_{V \supseteq f(U)} \mathcal{G}(V)$. If $\mathcal{G}$ is a sheaf of $\mathcal{O}_Y$-modules, then $f^{-1}\mathcal{G}$ is not a sheaf of $\mathcal{O}_X$-modules in any reasonable way, but $f^\#$ determines a map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ and the **pull-back**:

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

is rigged to be a sheaf of $\mathcal{O}_X$-modules. In the special case where $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, then $f^*(\widetilde{N}) = \widetilde{N \otimes_B A}$, and it follows that f^* commutes with tensor products (note that f_* does not!), and (quasi)-coherent sheaves pull back to (quasi)-coherent sheaves.

An S-scheme X is **separated** if $\Delta : X \to X \times_S X$ is a closed immersion. In that case, let $\mathcal{I}$ be the ideal sheaf of the diagonal. Then $\Omega_{X/S} := \Delta^*(\mathcal{I}/\mathcal{I}^2)$ is the **sheaf of relative differentials** of X over S. It is quasi-coherent by our discussion thus far, but this can be seen directly. Namely, for each $\mathrm{Spec}(B) \subset S$ and $\mathrm{Spec}(A) \subset X$ lying over $\mathrm{Spec}(B)$, the sheaf $\Omega_{X/S}$ restricts to $\widetilde{\Omega}_{A/B}$ on $\mathrm{Spec}(A)$, where $\Omega_{A/B}$ is the module of Kähler differentials for the associated map $\phi : B \to A$. If in addition S is Noetherian and X is an S-scheme **of finite type**, meaning that each $\pi^{-1}(\mathrm{Spec}(B))$ as above is covered by finitely many affines $\mathrm{Spec}(A)$ such that each A is a finitely generated B-**algebra**, then $\Omega_{X/S}$ is coherent because the $\Omega_{A/B}$ are finitely generated A-**modules** . A k-variety satisfies these assumptions, by definition, and if X is a non-singular k-variety, then $\Omega_X := \Omega_{X/\mathrm{Spec}(k)}$ is a locally free coherent sheaf of constant rank equal to the dimension of X, and vice-versa.

A morphism $f : X \to S$ is **proper** if X is separated and of finite type as an S-scheme and if in addition the projection maps $\pi_Y : X \times_S Y \to Y$ to all other S-schemes Y take closed sets to closed sets. Closed immersions and projective morphisms are proper, the latter by a theorem of Grothendieck.

A morphism $f : X \to S$ is **finite** if the inverse image of an affine open subscheme $\mathrm{Spec}(B) \subset S$ is always affine (call it $\mathrm{Spec}(A) \subset X$) and A is a finitely generated B-**module**. Finite morphisms are proper and quasi-finite, meaning that the inverse image of a point in S is always a finite set.

If $f : X \to S$ is proper and S is Noetherian, then coherent sheaves on X push forward to coherent sheaves on S (This is non-trivial to prove!) In particular, if $\mathcal{F}$ is a coherent sheaf on a projective k-scheme X, then the space $\Gamma(X, \mathcal{F})$ of global sections is a finite-dimensional vector space over k.

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on a scheme X, then $\Gamma(X, \mathcal{F})$ is a $\Gamma(X, \mathcal{O}_X)$-module by definition. We say $\mathcal{F}$ is **spanned** by a collection $\{s_i\}$ of sections of $\mathcal{F}$ if the images of the s_i generate every stalk of $\mathcal{F}$. Alternatively, the s_i determine a map $\oplus \mathcal{O}_X \to \mathcal{F}$, and $\mathcal{F}$ is spanned by the sections exactly when this map of sheaves of $\mathcal{O}_X$-modules is surjective.

The **fiber** of $\mathcal{F}$ at x is the vector space:

$$\mathcal{F}(x) := \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$$

For example, the fiber $\Omega_X(x)$ of a proper k-scheme X at a k-rational point is the cotangent space m_x/m_x^2, so that the sheaf Ω_X can be seen as the cotangent sheaf, even if X is singular.

It follows from Nakayama's lemma that if a map $\mathcal{F} \to \mathcal{G}$ of coherent sheaves induces a surjective map $\mathcal{F}(x) \to \mathcal{G}(x)$ at a point $x \in X$, then the sheaf map is surjective on stalks at x (hence in a neighborhood of x). In particular, the fiber dimension of a coherent sheaf $\mathcal{F}$ is an upper-semi-continuous function on X and $\mathcal{F}$ is generated by sections $\{s_i\}$ if and only each fiber $\mathcal{F}(x)$ is generated by the sections.

A coherent sheaf $\mathcal{L}$ is **invertible** if it is locally free of rank one. The dual sheaf $\mathcal{L}^* := Hom_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ inverts $\mathcal{L}$ in the sense that $\mathcal{L} \otimes \mathcal{L}^* \cong \mathcal{O}_X$ hence the terminology. Isomorphism classes of invertible sheaves form an abelian group, the **Picard group** $Pic(X)$, under tensor product.

Another Example. Suppose M is a graded module over $R := A[x_0, ..., x_n]$. There is a natural sheaf $\widehat{M}$ of $\mathcal{O}_{\mathbf{P}^n_A}$-modules on $\mathbf{P}^n_A$ with the property that $\widehat{M}|_{U_f} \cong \widetilde{M_{(f)}}$ for each "basic" open affine. This exhibits $\widehat{M}$ is a quasi-coherent sheaf, which is coherent if A is Noetherian and M is finitely generated.

In particular, if $M = \oplus M_e$ is a graded R-module, let $M(d)$ denote the "shifted" graded R-module with $M(d)_e = M_{d+e}$. In particular, the sheaves $\mathcal{O}_{\mathbf{P}^n_A}(d) := \widehat{R(d)}$ are invertible, and satisfy $\widehat{M(d)} \cong \widehat{M} \otimes_{\mathcal{O}_{\mathbf{P}^n_A}} \mathcal{O}_{\mathbf{P}^n_A}(d)$ for all graded modules M.

This time, the assignment $M \mapsto \widehat{M}$ does not give an equivalence between the categories of graded R-modules and of quasi-coherent sheaves on $\mathbf{P}^n_A$ because graded modules which are isomorphic in all sufficiently high degrees give rise to the same sheaves. This is the only ambiguity, however, and a sort of inverse is given by the twisted global sections:

$$\widehat{M} \mapsto \Gamma_* \widehat{M} := \bigoplus_{d \in \mathbf{Z}} \Gamma(\mathbf{P}^n_A, \widehat{M}(d))$$

If $\mathcal{F}$ is a quasi-coherent sheaf on $\mathbf{P}^n_A$, then $\mathcal{F} \mapsto \Gamma_* \mathcal{F} \mapsto \widehat{\Gamma_* \mathcal{F}}$ yields a quasi-coherent sheaf which is isomorphic to $\mathcal{F}$. It follows for example that closed subschemes of $\mathbf{P}^n_A$ correspond to homogeneous ideals in R.

If S is a scheme, then the invertible sheaves $\mathcal{O}_{\mathbf{P}^n_A}(d)$ over open affines $\mathrm{Spec}(A) \subset S$ glue to an invertible sheaf $\mathcal{O}_{\mathbf{P}^n_S}(d)$. An invertible sheaf $\mathcal{L}$ on an S-scheme X is **ample** (or S-ample) if there is a closed immersion $f : X \to \mathbf{P}^n_S$ (so X is projective, in particular) and a positive integer d such that $\mathcal{L}^{\otimes d} \cong f^* \mathcal{O}_{\mathbf{P}^n_S}(1)$. $\mathcal{L}$ is **very ample** if d may be chosen to be 1.

Any S-morphism $f : X \to \mathbf{P}^n_S$ determines $n+1$ global sections $s_0, ..., s_n$ generating the invertible sheaf $\mathcal{L} := f^* \mathcal{O}_{\mathbf{P}^n_S}(1)$ on X. These sections are pulled back from the "coordinate functions" $x_0, ..., x_n$ on $\mathbf{P}^n_S$. Conversely, an invertible sheaf $\mathcal{L}$ on X together with $n+1$ generating sections $s_0, ..., s_n$ uniquely determine an S-morphism to $\mathbf{P}^n_S$.

Suppose X is a proper scheme over an algebraically closed field k with invertible sheaf $\mathcal{L}$ and generating sections $s_0, ..., s_n \in \Gamma(X, \mathcal{L})$ as above. Let $V \subset \Gamma(X, \mathcal{L})$ be the **linear series** (i.e. the subspace) generated by the s_i. Then the associated morphism $f : X \to \mathbf{P}^n_k$ is a closed immersion if V **separates points**, meaning that for each pair $p, q \in X$ of closed points there is an $s \in V$ such that $s \in m_p \mathcal{L}$ but $s \notin m_q \mathcal{L}$, and V **separates tangent directions**, meaning that for each closed point $p \in X$, the vector space $m_p \mathcal{L} / m_p^2 \mathcal{L}$ is spanned by the sections $s \in V$ that lie in $m_p \mathcal{L}$. This is proved using Nakayama's lemma again, and applies even to non-reduced schemes.

Serre's Theorem A: If X is a projective scheme over a Noetherian ring A, let $\mathcal{O}_X(1)$ denote a very ample invertible sheaf on X. Then there is a $d_{\mathcal{F}}$ for each coherent sheaf $\mathcal{F}$ on X so that $\mathcal{F}(d) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$ is generated by its global sections whenever $d \geq d_{\mathcal{F}}$.

Very ampleness of a line bundle is difficult to detect without knowing the space of global sections. Ampleness, however, is a much easier notion to work with, as evidenced by the following refinement of Serre's Theorem A.

Criterion for Projectivity: Suppose A is Noetherian and X is a proper A-scheme with invertible sheaf $\mathcal{L}$. Then $\mathcal{L}$ is ample (hence X is projective) **if and only if** there is a $d_{\mathcal{F}}$ for each coherent sheaf $\mathcal{F}$ so that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}$ is generated by its global sections whenever $d \geq d_{\mathcal{F}}$.

Cohomology: Global properties of coherent sheaves are encoded in their cohomology spaces. Some important vanishing theorems are listed in this brief introduction to the most important tool in algebraic geometry.

The **global section functor** Γ is left exact, meaning that if:

$$0 \to \mathcal{F}' \stackrel{f'}{\to} \mathcal{F} \stackrel{f''}{\to} \mathcal{F}'' \to 0$$

is a short exact sequence of sheaves of abelian groups, then:

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$$

is exact, but the last map may not be surjective. The cohomology groups $\mathrm{H}^i(X, \mathcal{F})$ are the "right derived functors" of the global section functor. In particular, $\mathrm{H}^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ and given a short exact sequence as above, there are (functorial) connecting homomorphisms:

$$\delta^i : \mathrm{H}^i(X, \mathcal{F}'') \to \mathrm{H}^{i+1}(X, \mathcal{F}')$$

so that the following sequence is long-exact:

$$\ldots \stackrel{\mathrm{H}^i(f')}{\to} \mathrm{H}^i(X, \mathcal{F}) \stackrel{\mathrm{H}^i(f'')}{\to} \mathrm{H}^i(X, \mathcal{F}'') \stackrel{\delta^i}{\to} \mathrm{H}^{i+1}(X, \mathcal{F}') \stackrel{\mathrm{H}^{i+1}(f')}{\to} \ldots$$

A Noetherian scheme is a Noetherian space, meaning that it satisfies the descending chain condition on closed subsets. The maximal length of such a chain of irreducible closed subsets is the (Noetherian) dimension of the space. Hence the relevance of the following vanishing theorem of Grothendieck.

Vanishing Theorem: If X is a Noetherian space and $i > \dim(X)$, then $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all sheaves $\mathcal{F}$ of abelian groups on X.

Another vanishing theorem of Serre gives the following:

Criterion for Affineness: If X is a Noetherian scheme, then the following are equivalent:

(i) X is affine.

(ii) $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ and all $i > 0$.

(iii) $\mathrm{H}^1(X, \mathcal{I}) = 0$ for all coherent sheaves (of ideals) $\mathcal{I} \subset \mathcal{O}_X$.

On a Noetherian separated scheme X, an intersection of open affines is again affine. Using this fact, Serre's vanishing theorem allows us to conclude that the cohomology groups $\mathrm{H}^i(X, \mathcal{F})$ of a quasi-coherent sheaf $\mathcal{F}$ on such an X can be computed as the Cech cohomology with respect to **any** open affine cover. For example, one can use the standard cover of $\mathbf{P}^n_A$ to check:

(a) $A[x_0, ..., x_n] \cong \Gamma_* \mathcal{O}_{\mathbf{P}^n_A} = \oplus \mathrm{H}^0(\mathbf{P}^n_A, \mathcal{O}_{\mathbf{P}^n_A}(d))$.

(b) $\mathrm{H}^i(\mathbf{P}^n_A, \mathcal{O}_{\mathbf{P}^n_A}(d)) = 0$ for all d and all $0 < i < n$.

(c) $\mathrm{H}^n(\mathbf{P}^n_A, \mathcal{O}_{\mathbf{P}^n_A}(-n-1)) \cong A$.

(d) For each d, there is a perfect pairing of free A-modules:

$$\mathrm{H}^0(\mathbf{P}^n_A, \mathcal{O}_{\mathbf{P}^n_A}(d)) \times \mathrm{H}^n(\mathbf{P}^n_A, \mathcal{O}_{\mathbf{P}^n_A}(-d-n-1)) \to \mathrm{H}^n(\mathbf{P}^n_A, \mathcal{O}_{\mathbf{P}^n_A}(-n-1)) \cong A$$

Serre's Theorem B: If X is a projective scheme over a Noetherian ring A, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Then for each coherent sheaf $\mathcal{F}$, the cohomology groups $\mathrm{H}^i(X, \mathcal{F})$ are all finitely generated A-modules, and there is a $d_{\mathcal{F}}$ such that $\mathrm{H}^i(X, \mathcal{F}(d)) = 0$ for each $d \geq d_{\mathcal{F}}$ and $i > 0$.

Once again, this is refined by replacing "very ample" with "ample."

Criterion for Projectivity (2): Suppose A is Noetherian and X is a proper A-scheme with invertible sheaf $\mathcal{L}$. Then $\mathcal{L}$ is ample **if and only if** there is a $d_{\mathcal{F}}$ for each coherent sheaf $\mathcal{F}$ so that $\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes d}) = 0$ whenever $i > 0$ and $d \geq d_{\mathcal{F}}$.

Remark: Note that the criteria are independent of the Noetherian ring A. Thus ampleness, unlike very ampleness, only depends upon X, and not the map to $\mathrm{Spec}(A)$. I.e. ampleness is an "absolute" property of X.

Let X be a projective scheme over a field k. Then the **Hilbert function**:

$$d \mapsto \dim_k \Gamma(X, \mathcal{F}(d))$$

is polynomial in d for sufficiently large d. This polynomial $P_{\mathcal{F}}(d)$ is the **Hilbert polynomial** of $\mathcal{F}$. We can see the polynomial growth of the Hilbert function in one of two ways.

The **projection formula** states that if $f : X \to Y$ is a morphism of locally ringed spaces and $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_Y$-modules, then:

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}$$

If we apply this to the closed immersion $f : X \to \mathbf{P}^n_k$ with $\mathcal{E} = \mathcal{O}_{\mathbf{P}^n_k}(d)$, we get:

$$\dim_k \Gamma(X, \mathcal{F}(d)) = \dim_k \Gamma(\mathbf{P}^n_k, f_*\mathcal{F} \otimes \mathcal{O}_{\mathbf{P}^n_k}(d))$$

It follows from Serre's theorem A that the graded module $M := \widehat{\Gamma_* f_*\mathcal{F}}$ over $k[x_0, ..., x_n]$ is finitely generated, therefore the dimensions of the M_d grow as a polynomial in d for sufficiently large d. (See e.g. Atiyah-Macdonald)

Alternatively, we may define the **Euler characteristic** of $\mathcal{F}(d)$ by:

$$\chi(X, \mathcal{F}(d)) = \sum_{i=0}^{\dim(X)} (-1)^i \dim_k \mathrm{H}^i(X, \mathcal{F}(d))$$

Then $\chi(X, \mathcal{F}(d)) = \dim_k \Gamma(X, \mathcal{F}(d))$ for all sufficiently large d by Serre's theorem B. Thus it suffices (and gives us much more information!) to prove that the Euler characteristic $\chi(X, \mathcal{F}(d))$ is polynomial in d for all d, hence equal to the Hilbert polynomial. This follows from the fact that the Euler characteristic is additive on exact sequences. If $X_1, ..., X_m \subset X$ are the irreducible components of maximal dimension, then multiplication by any section $s \in \mathcal{O}_X(1)$ that is nonzero on each of these components gives rise to an exact sequence:

$$0 \to i_*\mathcal{N} \to \mathcal{F} \xrightarrow{\cdot s} \mathcal{F}(1) \to i_*\mathcal{Q} \to 0$$

where $\mathcal{N}$ and $\mathcal{Q}$ are coherent sheaves on the zero scheme $D \subset X$ of s. Since $\dim(D) = \dim(X) - 1$, and since tensoring by $\mathcal{O}_X(d)$ is exact, the polynomiality follows by induction on dimension, the projection formula, and the observation that cohomology commutes with closed immersions. We also see in this way that the degree of the Hilbert polynomial is bounded above by the dimension of X.