Mathematics 43000 – Foundations of Algebra – Fall 1999

5. A Tour of Numbers (The End of the Line). Our tour has taken us through the natural numbers, integers, rational numbers, real numbers and complex numbers. It is certainly reasonable to wonder whether we can continue in this way forever, discovering new number systems with ever better properties. However it turns out that we cannot, in the following sense.

Theorem: Other than $\mathbf{C} = \mathbf{R}^2$ and $\mathbf{R}$ itself, there is no way to put a multiplication on $\mathbf{R}^n$ so that it becomes a field.

Remark: There are infinite-dimensional vector spaces which are also fields. But those should be thought of as spaces of functions rather than of numbers.

We will look closely at this theorem later. In the meantime, we'll finish the tour with some "near misses"... multiplications on vector spaces which do not quite produce fields. This is not meant to be an exhaustive list of such things. But it does give the general sense of what goes wrong.

The Cross Product on $\mathbf{R}^3$: Recall the cross product formula:

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = (y_1z_2 - z_1y_2, z_1x_3 - x_1z_3, x_1y_2 - y_1x_2)$$

which I always remember in the following way. Let:

$$i = (1, 0, 0),\ j = (0, 1, 0),\ \text{and } k = (0, 0, 1)$$

then

$$i \times j = k, j \times k = i, k \times i = j$$

and the cross product is anti-commutative: $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$ for all $\vec{v}, \vec{w} \in \mathbf{R}^3$ so that in particular $\vec{v} \times \vec{v} = 0$ always.

The cross product is introduced because of its useful geometric properties. Namely if $\vec{v}$ and $\vec{w}$ are vectors in $\mathbf{R}^3$ which are not scalar multiples of each other, then there is an definite angle θ between them, measured in the plane spanned by $\vec{v}$ and $\vec{w}$, and

$$\vec{v} \times \vec{w}$$

is the unique vector perpendicular to both $\vec{v}$ and $\vec{w}$ with length $|\sin(\theta)||\vec{v}||\vec{w}|$ and pointing in the direction determined by the "right-hand-rule"
(Sound familiar? Check any book on calculus in more than one variable.)

This is a pretty compelling product, and it looks superficially like a good three-dimensional version of the multiplication of complex numbers. But it is very, very far from giving us a field! To see this, let's check the properties of a field from the tour of the rational numbers.

(a) Vector addition is always associative.

(b) Vector addition always has the origin as the additive identity.

(c) $-\vec{v}$ is always the additive inverse of $\vec{v}$.

(d) Vector addition is always commutative.

Remark: Since (a)-(d) are always true for vector spaces, we won't mention them again in this section.

(e) **The cross product is not associative!**

Example: $i \times (i \times j) = i \times k = -j$ but $(i \times i) \times j = 0 \times j = 0$.

(f) The cross product does distribute with addition. (Exercise)

(g) **The cross product does not have a multiplicative identity!**

Proof: If there were a multiplicative identity, we would have $1 \times 1 = 1$. But anything crossed with itself is zero! And zero crossed with anything is zero.

(h) It doesn't even make sense to ask for a multiplicative inverse! (Why?)

(i) **The cross product is not commutative!**

Example: $i \times j = k$ but $j \times i = -k$.

So this seems to be a real loser. Why then do we call it a product at all? Because it satisfies (f). Membership in the club of products is not exclusive.

Remark: You may remember that there was one other product in $\mathbf{R}^3$, namely the dot product. The formula is:

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = x_1x_2 + y_1y_2 + z_1z_2$$

but this product is **totally unacceptable** as a multiplication because it doesn't even produce a vector! It is, however, distributive, which is why it claims membership in the club of products.

Now we move on to the closest near-miss, which takes place in $\mathbf{R}^4$.

The Quaternions: We declare **three** square-roots of -1, namely

$$i^2 = -1, j^2 = -1, k^2 = -1$$

and we use the same letters as in the cross-product because these are declared to have the same products with **each other** as in the cross product. Thus,

$$ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j$$

We set up 4-space with coordinates t, x, y, z (and if you balk at this, let me remind you that we **live** in such a vector space. We locate the real numbers on the t-axis, so that:

$$0 = (0,0,0,0) \text{ and } 1 = (1,0,0,0)$$

and we locate i, j, k on the other axes:

$$i = (0,1,0,0), j = (0,0,1,0) \text{ and } k = (0,0,0,1)$$

We define the **quaternions**:

$$\mathbf{H} := \{\text{points in 4-space}\}$$

with vector addition, and multiplication determined by the distributive law and the products listed above (just as multiplication of complex numbers was determined by the distributive law and $i^2 = -1$).

Remark: Why **H**? **Q** was taken by the rationals (and **R** by the reals). But more importantly, the quaternions were first discovered by a British physicist named Hamilton who was apparently so excited by this discovery that he carved the rules for the product into a bridge. In deference to Hamilton, we use **H** to denote the quaternions.

Here is the formula:

$$(a,b,c,d) \cdot (t,u,v,w) =$$

$$(at - bu - cv - dw, au + bt + cw - dv, av + ct + du - bw, aw + dt + bv - cu)$$

Given the miserable performance of the cross product, this trick of setting $i^2 = j^2 = k^2 = -1$ (instead of 0) has some miraculous consequences!

(e) **Multiplication of quaternions is associative**

You are cordially invited to check this with the formula. (And you should indeed do this once in your life, if only to see why Hamilton got so excited.) This is the sort of check that one should carry out in private. Let me show you how the counter-example for the cross product works out here(!)

$$i \cdot (i \cdot j) = i \cdot k = -j \text{ exactly as before, but}$$

$$(i \cdot i) \cdot j = (-1)j = -j \quad \text{which now matches!}$$

(f) Quaternion multiplication distributes with addition. (Exercise)

(g) 1 **is the multiplicative identity**

$$(1, 0, 0, 0) \cdot (t, u, v, w) = (1t, 1u, 1v, 1w) = (t, u, v, w)$$

by the formula. And what's more:

(h) **Every quaternion except zero has a multiplicative inverse**

We accomplish this with **quaternion conjugation**:

$$\overline{t + ui + vj + wk} = t - ui - vj - wk$$

and just as in the case of the complex numbers, we obtain:

$$(t + ui + vj + wk)\overline{(t + ui + vj + wk)} = t^2 + u^2 + v^2 + w^2$$

which is a positive real number whenever $t + ui + vj + wk \neq 0$, so

$$\left(\frac{1}{t^2 + u^2 + v^2 + w^2}\right)(t - ui - vj - wk) = (t + ui + vj + wk)^{-1}$$

is the multiplicative inverse to $(t + ui + vj + wk)$.

Alas:

(i) **Multiplication of quaternions is not commutative.**

After all, we still have:

$$i \cdot j = -j \cdot i$$

as in the case of the cross product.

The quaternions do turn out to be useful in some areas of mathematics, so this has not been an arbitrary exercise. Also, they are unusual, in the sense that having multiplicative inverses in a non-commutative multiplication is a very rare phenomenon. (One can achieve this in an 8-dimensional vector space, but at the cost of losing associativity, too.)

While we are at it, we really should consider:

The Naive Multiplication on $\mathbf{R}^n$:

$$(s_1, ..., s_n)(t_1, ..., t_n) = (s_1t_1, ..., s_nt_n)$$

coordinate by coordinate, as with vector addition. Then it is obvious that:

(e) **Naive multiplication is associative**

(f) **Naive multiplication distributes with addition**

(g) $(1, 1, 1, ..., 1)$ **is the multiplicative identity**, but

(h) Not every element other than zero has a multiplicative inverse.

The only possible candidate for the multiplicative inverse to $(s_1, ..., s_n)$ is:

$$(\frac{1}{s_1}, ..., \frac{1}{s_n})$$

and this works, provided **none of the coordinates is zero**. If there were a zero in one of the coordinates, we'd be dividing by zero, and there is no way to rectify that problem.

(i) **Naive multiplication is commutative.**

So in a way, this was a near-miss, too. The quaternions are usually considered a closer near-miss because having multiplicative inverses allows one to divide, even when multiplication is not commutative(!)

Exercises:

5.1 (a) Show that $\vec{v} \times \vec{v} = 0$ for all vectors $\vec{v}$ in $\mathbf{R}^3$.

(b) Show that the cross product distributes with addition.

5.2 Show that quaternion multiplication is associative.

5.3 (a) Calculate $(ui + vj + wk)^2$ with quaternion multiplication.

(b) Since multiplication of quaternions is not commutative, one needs to be careful about what we mean by a multiplicative inverse. Check that:

$$(t+ui+vj+wk)(t+ui+vj+wk)^{-1} = 1 = (t+ui+vj+wk)^{-1}(t+ui+vj+wk)$$

in other words, although $\mathbf{H}$ is not commutative in general, the multiplicative inverse is the same on either side. (This is very important.)

5.4 Let's investigate another near-miss.

Matrix Multiplication on $\mathbf{R}^4$. We think of the coordinates of the vector space $\mathbf{R}^4$ as living in a 2×2 matrix:

$$\mathbf{R}^4 = \{\begin{pmatrix} s & t \\ u & v \end{pmatrix}\}$$

and notice that matrix addition is the same as vector addition. But matrix multiplication is one that we have not seen before. Namely:

$$\begin{pmatrix} s & t \\ u & v \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} sw + ty & sx + tz \\ uw + vy & ux + vz \end{pmatrix}$$

(a) Show that matrix multiplication is associative.

(b) Show that matrix multiplication distributes with addition.

(c) Find the multiplicative identity.

(d) Explain exactly which matrices have multiplicative inverses, find the multiplicative inverse, and check as in 5.3 that it is the same on either side.

(e) Show that matrix multiplication is not commutative.