Mathematics 4300 – Foundations of Algebra – Fall 1999

3. Next stop on the tour:

The Real Numbers: Let $\mathbf{R}$ denote the set of real numbers:

$$\mathbf{R} = \{\text{points on the number-line}\}$$

and let $\mathbf{R}^+$ denote the set of positive real numbers:

$$\mathbf{R}^+ = \{\text{points to the right of zero}\} = \{\text{lengths of line segments}\}$$

First Question: Are all real numbers rational?

Answer: No. For example, $\sqrt{2}$, which is the length of the hypotenuse of a right triangle with side lengths 1 and 1, is not rational. Here's the proof.

Suppose $\sqrt{2}$ were rational. Then we could write it in the form $\sqrt{2} = \frac{m}{n}$ (because $\sqrt{2}$ is positive). If we square both sides and multiply by n^2, we'd have $2n^2 = m^2$. On the other hand, the fundamental theorem of arithmetic (from §1) tells us that m, n, m^2, n^2 each factor uniquely as products of primes. Take the factorization for m and double all the exponents. This must therefore give the factoriztion for m^2, and similarly for n and n^2. This tells us that the exponent of 2 in the factorization for m^2 must be **even**. But the exponent of 2 in the factorization of $2n^2$ must be **odd** by the same reasoning, so it is not possible to have $2n^2 = m^2$, and $\sqrt{2}$ is therefore not rational.

The same reasoning shows that $\sqrt{m}$ is never rational unless m is a "perfect square", and similarly the nth roots $\sqrt[n]{m}$ are never rational unless m is an nth power of a natural number. It is harder to show that other familiar numbers, such as π (the ratio of the circumference of a circle to its diameter) and e (the base of the natural logarithm) are not rational, but they are not.

We will be able to find lots more irrational numbers using:

Decimal Notation: A decimal is given by:

$$q.d_1d_2d_3d_4.... \quad \text{or} \quad -(q.d_1d_2d_3d_4...)$$

where q is a natural number or zero, and each d_i is between 0 and 9. We will adopt the convention that all decimals are infinite. A **terminating decimal** is an infinite one which ends with 0000.....

Procedure for associating a decimal to a positive real number:

If a positive real number r is given, let q be the largest integer so that:

$$q \leq r$$

(note that q is natural or zero) and let d_1 be the largest integer so that:

$$q + \frac{d_1}{10} \leq r$$

(note that $0 \leq d_1 \leq 9$ because of the choice for q) If we have carried out the procedure to m steps:

$$q + \frac{d_1}{10} + \frac{d_2}{100} + ... + \frac{d_m}{10^m} \leq r$$

where each of $q, d_1, ..., d_m$ is as large as possible, then let d_{m+1} be as large as possible so that:

$$q + \frac{d_1}{10} + \frac{d_2}{100} + ... + \frac{d_m}{10^m} + \frac{d_{m+1}}{10^{m+1}} \leq r$$

Since we refuse to stop even if we hit r on the nose, we always "obtain" an infinite decimal in this way.

We say the real number r is **represented** by the decimal $q.d_1d_2d_3....$

A non-obvious observation: We never obtain a decimal ending in 99999...

For example, suppose we got 0.999999.... out of some real number r. Then $r < 1$, since otherwise we would have begun the decimal expansion with a "1". On the other hand, having all 9's in the decimal expansion tells us that $\frac{9}{10} < \frac{99}{100} < \frac{999}{1000} < ... < r$ and since this sequence of (rational) numbers has a limit of 1, we know that it eventually surpasses every number less than 1, and so it **cannot** be true that $r < 1$. (Calculus has reared its head here!) Thus our real number r must be less than 1 and at the same time cannot be less than 1. There is no such number! So we never get 0.9999....

The proof of the observation in general follows the same lines. If we had a decimal expansion of the form $q.d_1d_2...d_m99999...$ then r would be, at the same time, less than $q.d_1d_2...d_m + 10^{-m}$ and greater than it. So there is no such number.

Out of the decmial expansion for r, we have an increasing sequence:

$$q, \quad q + \frac{d_1}{10}, \quad q + \frac{d_1}{10} + \frac{d_2}{100}, \ ...$$

of rational numbers with the real number r as its limit. (Calculus again!) Since different decimals have different limits, we see that each positive real number has a unique decimal expansion, that is, we get a **bijection** between the (huge!) sets:

$$\mathbf{R}^+ \leftrightarrow \{\text{decimals not ending in } 9999.....\}$$

Examples:

(i) Natural numbers. When applied to a natural number m, the procedure above yields: $m.000000....$

(ii) $\sqrt{2}$. We can do this by trial and error!

$1^2 < 2$ and $2^2 > 2$, so $q = 1$.

$(1.4)^2 < 2$ and $(1.5)^2 > 2$, so $d_1 = 4$.

$(1.41)^2 < 2$ and $(1.42)^2 > 2$, so $d_2 = 1$

$(1.414)^2 < 2$ and $(1.415)^2 > 2$, so $d_3 = 4$

etc

Thus $\sqrt{2} = 1.414...$ (We should interpret the "..." as "to be continued.")

(iii) The positive rational numbers. If $r = \frac{m}{n}$, then we obtain the decimal by repetition of division with remainders. Namely:

$m = nq + r_1$ (here we allow the possibility $q = 0$)

$10r_1 = nd_1 + r_2$ (division with remainders again)

$10r_2 = nd_2 + r_3$ (and again)

$10r_3 = nd_3 + r_4$ (and again)

etc.

Notice that since each remainder $r_1, r_2, ...$ is less than n and non-negative, it follows that each $d_1, d_2, ...$ is at most 9, and at least 0. In other words, $q.d_1d_2d_3....$ is a decimal expansion. By substituting, we get:

$$\frac{m}{n} = q + \frac{r_1}{n} = q + \frac{d_1}{10} + \frac{r_2}{10n} = q + \frac{d_1}{10} + \frac{d_2}{100} + \frac{r_3}{100n} = ...$$

For example, dividing the first equation by n gives:

$$\frac{m}{n} = q + \frac{r_1}{n}$$

and dividing the second by $10n$ gives:

$$\frac{r_1}{n} = \frac{d_1}{10} + \frac{r_2}{10n}$$

which we substitute for $\frac{r_1}{n}$ to get the second equality, etc.

Remark: Example (iii) is the method we learned in school for finding the decimal expansion of a rational number! The last paragraph, which is often not taught, explains **why** the decimal expansion and the fraction represent the same point on the number-line(!)

Negatives: If r positive real number, then $-r$ is the negative real number located r units to the left of 0. As with the integers, $-(-r) = r$ and $-0 = 0$.

Now for the operations of addition and multiplication. It is a little tricky to add and multiply infinite decimals. You can do it if you are very careful about significant figures, but we will use a more geometric approach.

Addition: The sum $s+t$ of two real numbers is defined as before by moving right or left on the number-line starting at s. Then just as before,

Addition is associative and commutative,

Zero is the additive identity, and $-s$ is the additive inverse of s

Subtraction is defined as always, as addition with the additive inverse.

Relations are defined in terms of position on the number-line as before.

Multiplication: We'll define multiplication by geometry, in terms of areas. If r and u are positive real numbers, then ru is the area of an $r \times u$ rectangle. Note that an area only represents a point on the number-line using calculus, by filling the rectangle with as many $\frac{1}{n} \times \frac{1}{n}$ squares as possible, counting them, and "taking the limit". The rule for multiplying arbitrary real numbers is then the same as the rule for integers:

$$(-r)u = -(ru), \quad r(-u) = -(ru) \quad \text{and} \quad (-r)(-u) = ru$$

and of course 0 times anything is 0.

Just as when we counted arrays for natural numbers, the basic properties for multiplication of positive real numbers are clear from geometry. For example, $(ru)v = r(uv)$ because both are the **volumes** of an $r \times u \times v$ box. And as we did with the integers, we can include the rules for multiplying negatives to conclude that:

Multiplication is commutative, associative, distributive, and

One is the multiplicative identity

It is not as easy to see that every real number has a multiplicative inverse. To do it, we'll use a theorem from analysis. Let r be a positive real number. Then the function:

$$f(x) = r \cdot x$$

is a continuous function (in fact, differentiable with derivative $f'(x) = r$). Since $f(0) = 0$ and $\lim_{x\to+\infty} f(x) = +\infty$, the **intermediate value theorem** tells us that there must be some positive real number s so that $f(s) = 1$. In other words, $rs = 1$, and so s is the multiplicative inverse to r. We will forever more write the multiplicative inverse to r as $\frac{1}{r}$ or r^{-1}.

Of course, once we have a multiplicative inverse to positive real numbers, then we have a multiplicative inverse to negative ones, by setting $\frac{1}{-r} = -\frac{1}{r}$. But this tells us that:

R is a field!

I said that multiplying and adding decimals is tricky. There are a couple of useful exceptions to this.

Multiplying by a power of 10: Given the decimal: $q.d_1d_2d_3...$ then:

$$10(q.d_1d_2d_3...) = (10q + d_1).d_2d_3d_4...$$

$$100(q.d_1d_2d_3...) = (100q + 10d_1 + d_2).d_3d_4d_5...$$

etc.

Thus $10(5.33333...) = 53.33333...$ and $100(5.3333...) = 533.3333...$ so that multiplying by powers of 10 simply shifts the decimal.

Subtracting "matching" decimals:

$$q.d_1d_2d_3... - p.d_1d_2d_3... = q - p$$

A **repeating** decimal is one which has the form:

$$q.d_1d_2...d_m...d_nd_m...d_nd_m...d_n...$$

in which case we write it as: $q.d_1d_2...\overline{d_m...d_n}$

Example:

$$\frac{1}{35} = 0.0\overline{285714}$$

Proposition: Repeating decimals are rational numbers and vice versa.

Proof: We have to prove two things. Starting with a repeating decimal, we have to show that it is a rational number. And starting with a rational number, we have to show that it is a repeating decimal.

So suppose we are given $r = q.d_1d_2...\overline{d_m...d_n}$. Then:

$$10^{m-1}r = (10^{m-1}q + 10^{m-2}d_1 + ... + d_{m-1}).\overline{d_m...d_n}$$

and

$$10^{n-1}r = (10^{n-1}q + 10^{n-2}d_1.... + d_{n-1}).\overline{d_m...d_n}$$

are matching decimals, so we can subtract them to get:

$$10^{n-1}r - 10^{m-1}r = (10^{n-1}q + ... + d_{n-1}) - (10^{m-1}q + ... + d_{m-1})$$

and dividing both sides by $10^{n-1} - 10^{m-1}$, we get:

$$r = \frac{(10^{n-1}q + ... + d_{n-1}) - (10^{m-1}q + ... + d_{m-1})}{10^{n-1} - 10^{m-1}}$$

which is a rational number!

Example: If $r = 5.34\overline{57}$, then:

$$100r = 534.\overline{57} \text{ and } (10,000)r = 53457.\overline{57}$$

so

$$r = \frac{53457 - 534}{10,000 - 100}$$

We're half-way done with the proof! For the second half, suppose we start with a fraction $\frac{m}{n}$. We need to know that it gives a repeating decimal.

Let's go back to the earlier example (Example (iii)) where we produced the decimal expansion of the rational number $\frac{m}{n}$. The was done by infinitely many divisions with remainders of the form:

$$10r_i = nd_i + r_{i+1}$$

Notice that all the remainders $r_1, r_2, r_3, ...$ are trapped between 0 and $n-1$. Thus we know that many of them (in fact infinitely many) must be the same. Choose two of these which are the same and call them r_i and r_j (and we may suppose $i < j$). Then because of the equations:

$$10r_i = nd_i + r_{i+1} \text{ and } 10r_i = 10r_j = nd_j + r_{j+1}$$

and because division with remainders produces a **unique** quotient and remainder, we know that $d_i = d_j$ and also that $r_{i+1} = r_{j+1}$, so that repeating the argument gives $d_{i+1} = d_{j+1}$, etc. Thus the decimal **repeats**:

$$\frac{m}{n} = q.d_1 d_2 ... \overline{d_i ... d_{j-1}}$$

Example: Converting $\frac{552}{550}$ to a decimal

$551 = 550(1) + 2$ (so $q = 1$ and $r_1 = 2$).

$10(2) = 550(0) + 20$ (so $d_1 = 0$ and $r_2 = 20$...no repeats)

$10(20) = 550(0) + 200$ (so $d_2 = 0$ and $r_3 = 100$...no repeats)

$10(200) = 550(3) + 350$ (so $d_3 = 3$ and $r_4 = 350$...no repeats)

$10(350) = 550(6) + 200$ (so $d_4 = 6$ and $r_5 = 200$...repeat! $d_3 = d_5$, so)

$$\frac{552}{550} = 1.00\overline{36}$$

So there are more real numbers than rational numbers: Just pick your favorite non-repeating decimal. It will give a real number which is not rational. My favorite is:

$$1.0100100010000100001.....$$

which has a definite pattern, but not a repeating one!

A Final Remark: We've shown in this section that $\sqrt{2}$ is not rational, and that repeating decimals are rational. We can conclude from this that the decimal for $\sqrt{2}$ never repeats. For instance, it is **not** $1.\overline{41}$ as might have been suggested by our small number of calculations.

Exercises:

3.1 (a) Prove that $\sqrt[3]{2}$ is not rational.

(b) Prove that $\sqrt[n]{2}$ is not rational for any $n > 1$.

3.2 Convert the following into decimals (Do it without a calculator and only use the calculator to check your work!):

(a) $\frac{1}{7}$

(b) $\frac{1}{11}$

(c) $\frac{1}{13}$

(d) $\frac{1}{17}$ (this doesn't look like it repeats on the calculator!)

Do you notice any pattern in the lengths of the repeated parts?

3.3 Convert the following repeating decimals to fractions.

(a) $0.\overline{1}$

(b) $0.\overline{01}$

(c) $0.\overline{10}$

(d) $0.0\overline{01}$

3.4 Suppose that $\frac{1}{n}$ converts to a decimal of the form $0.\overline{d_1 d_2}$.

Show that n is either $3, 9, 11, 33$ or 99.

3.5 When $n > 1$, is it possible that:

$$\frac{1}{n} = 0.\overline{d_1 d_2 ... d_n}$$

and this is the smallest pattern to repeat? If so, why? If not, why not?

(For example, $\frac{1}{3} = 0.\overline{333}$ but this is not the smallest pattern, $0.\overline{3}$ is.)

3.6 Suppose $\frac{1}{n}$ is represented by a terminating decimal. What can you say about n?

Food for thought: How might you go about finding the first fifty terms in the decimal expansion for π? For e? (Without looking them up!)