Mathematics 4300 – Foundations of Algebra – Fall 1999

11. Formulas for Roots. The fundamental theorem of algebra tells us that every polynomial has complex roots, but it doesn't tell us how to find them! In the case of quadratic, cubic and quartic polynomials, there are formulas for the roots, which we will discuss in this section. The quadratic formula is obtained by completing the square, but the others require more "trickery".

The Quadratic Formula: We wish to solve the equation:

$$az^2 + bz + c = 0$$

Preliminary Step: Divide through by a. We get:

$$z^2 + \frac{b}{a}z + \frac{c}{a} = 0$$

The Main Idea: Complete the square. This gives:

$$z^2 + \frac{b}{a}z + \frac{c}{a} = (z + \frac{b}{2a})^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right) = 0$$

Let $w = z + \frac{b}{2a}$. Then in terms of w, the roots are:

$$w = \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} = \frac{\sqrt{b^2 - 4ac}}{2a}$$

and recall that every complex number (except zero) has **two** square roots. In terms of the original variables and coefficients, this gives:

$$z = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Definition: $\Delta = b^2 - 4ac$ is the **discriminant** of $ax^2 + bx + c$.

Observation: From the quadratic formula, we see that:

(a) If $\Delta = 0$, then there is only one root, namely $z = \frac{-b}{2a}$. Otherwise, there are two roots.

(b) If a, b, c are **real** numbers and $\Delta < 0$, then there are no real roots. If $\Delta > 0$, then both roots are real.

Examples: (a) $z^2 + z + 1$ has roots:

$$z = \frac{-1 + \sqrt{-3}}{2}$$

so $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ are the two roots. They are not real.

(b) $z^2 + z + i$ has roots:

$$z = \frac{-1 + \sqrt{1 - 4i}}{2}$$

and the two values for $\sqrt{1 - 4i}$ can be calculated using polar coordinates. Namely, $1 - 4i = (\sqrt{17}; \tan^{-1}(-4))$ in polar coordinates, so

$$(\sqrt[4]{17}; \frac{\tan^{-1}(-4)}{2}) \approx 1.6 - 1.25i \quad \text{and} \quad (\sqrt[4]{17}; \frac{\tan^{-1}(-4)}{2} + \pi) \approx -1.6 + 1.25i$$

are the two square roots. Plugging this in to the formula gives the following approximate answers for the roots:

$$z \approx 0.3 - 0.625i \quad \text{or} \quad z \approx -1.3 + 0.625i$$

(note that these roots are not conjugates because of the i in the equation!)

Cardano's Cubic Formula: We wish to solve the equation

$$az^3 + bz^2 + cz + d = 0$$

Preliminary Step: Divide through by a. We get:

$$z^3 + \frac{b}{a}z^2 + \frac{c}{a}z + \frac{d}{a} = 0$$

First Step: Complete the **cube**. Thus:

$$z^3 + \frac{b}{a}z^2 + \frac{c}{a}z + \frac{d}{a} = (z + \frac{b}{3a})^3 + (\frac{c}{a} - \frac{b^2}{3a^2})(z + \frac{b}{3a}) + (\frac{d}{a} - \frac{bc}{3a^2} + \frac{2b^3}{27a^3})$$

Let $w = z + \frac{b}{3a}$, $p = \frac{c}{a} - \frac{b^2}{3a^2}$ and $q = \frac{d}{a} - \frac{bc}{3a^2} + \frac{2b^3}{27a^3}$. Then we need to solve:

$$w^3 + pw + q = 0$$

If we are very lucky and it happens that $p = 0$, then we get:

$$w = \sqrt[3]{-q}$$

for the **three** cube roots of $-q$, and in the original variables this gives:

$$z = \frac{-b + \sqrt[3]{9abc - 2b^3 - 27a^2 d}}{3a}$$

which looks a lot like the quadratic formula. If $p \neq 0$, though, we need the:

Second Step (An Amazing Substitution): Let $w = u - \frac{p}{3u}$. Then

$$0 = w^3 + pw + q = u^3 - \frac{p^3}{27u^3} + q$$

which we multiply through by u^3 to get:

$$u^6 + qu^3 - \frac{p^3}{27} = 0$$

We can use the quadratic formula on this(!) to get:

$$u^3 = \frac{-q + \sqrt{q^2 + \frac{4p^3}{27}}}{2} = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

so that

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Remembering that:

$$w = u - \frac{p}{3u} \quad \text{and} \quad z = w - \frac{b}{3a}$$

allows us to use this to find all the roots. (Actually, this gives us six solutions since there are two choices for the square root and three choices for the cube root...however it turns out that the choice of square root doesn't matter.)

Definition: $\Delta = \frac{q^2}{4} + \frac{p^3}{27}$ is the **discriminant** of $az^3 + bz^2 + cz + d$.

Note: Recall that $p = \frac{c}{a} - \frac{b^2}{3a^2}$ and $q = \frac{d}{a} - \frac{bc}{3a^2} + \frac{2b^3}{27a^3}$.

Example: Find the roots of $z^3 - z$.

Here $w = z$, and $p = -1$ and $q = 0$. So $w = u - \frac{p}{3u} = u + \frac{1}{3u}$ and:

$$u = \sqrt[3]{0 + \sqrt{0 + \frac{-1}{27}}} = \sqrt[6]{\frac{-1}{27}}$$

and the first three of these sixth roots are:

$$u = \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right), \; u = \frac{i}{\sqrt{3}} \text{ and } u = \frac{1}{\sqrt{3}}\left(-\frac{\sqrt{3}}{2} + \frac{\sqrt{1}}{2}i\right)$$

If we plug in for $w = u + \frac{1}{3u}$, we get:

$$w = 1, \; w = 0 \text{ and } w = -1$$

which we can check by factoring!

A Messier Example: Find the roots of $z^3 - z + 1$.

Here $w = z$ and $p = -1$ and $q = 1$. So again $w = u + \frac{1}{3u}$, and:

$$u = \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{27}}} \approx \sqrt[3]{-0.03852}$$

This gives (approximate) values:

$$u \approx 0.3377\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), \; u \approx -0.3377 \text{ and } u \approx 0.3377\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

which after we plug in for w give us:

$$w \approx .6624 - 0.5624i, \; w \approx -1.325, \; \text{ and } w \approx .6624 + 0.5624i$$

These roots are only approximate. Factoring is extremely difficult here!

Notice that in the first example there were three real roots while in the second, there was only one. The discriminant defined above plays a crucial role in determining whether there will be one or three, as we see from the following:

Proposition: Let $p(z) = az^3 + bz^2 + cz + d$, and $\Delta = \frac{q^2}{4} + \frac{p^3}{27}$.

(a) If $\Delta \neq 0$, then $p(z)$ has three different roots.

(b) If a, b, c, d are all real and $\Delta > 0$, then there are one real and two complex roots. If $\Delta < 0$, then all the roots are real.

Proof: Consider the polynomial

$$f(w) = w^3 + pw + q = 0$$

Since $z = w - \frac{b}{3a}$, we get the roots of $p(z)$ by subtracting $\frac{b}{3a}$ from the roots of $f(w)$. So we may as well prove the proposition for $f(w)$. Let r be a root of $f(w)$. Recall from your homework that $f(w)$ will be divisible by $(w-r)^2$ exactly when:

$$f'(w) = 3w^2 + p = 0$$

also vanishes at r. So we try to simultaneously solve:

$$f(r) = f'(r) = 0$$

and we do this in a clever way. If $f'(r) = 0$, then $p = -3r^2$, so:

$$f(r) = r^3 + pr + q = r^3 - 3r^3 + q = -2r^3 + q$$

But then $f(r) = 0$ happens when $q = 2r^3$. Putting these together, we get:

$$\frac{q^2}{4} = r^6 = -\frac{p^3}{27}$$

In other words, in order to solve, we must have $\Delta = 0$. This gives (a).

For (b), notice that if $p > 0$ (so that automatically, $\Delta > 0$), then $f(w)$ has no critical points, and it follows that there is only one real root (the derivative is always positive!). If $p = 0$, there is only one critical point, which still means there can only be one real root. If $p < 0$, then there are critical points at $s_1 = -\sqrt{\frac{-p}{3}}$ and $s_2 = \sqrt{\frac{-p}{3}}$. We will have three roots if $f(s_1) > 0$ and $f(s_2) < 0$ by the intermediate value theorem! But:

$$f(s_1) = \left(-\sqrt{\frac{-p}{3}}\right)^3 + p\left(-\sqrt{\frac{-p}{3}}\right) + q = \frac{-2p}{3}\sqrt{\frac{-p}{3}} + q$$

This is positive exactly when:

$$q > \frac{2p}{3}\sqrt{\frac{-p}{3}}$$

If q is positive, then this is automatic. If q is negative, then both sides are negative, so the inequality switches when we square both sides, and we get:

$$q^2 < \frac{4p^3}{27}$$

which happens exactly when $\Delta < 0$. Thus if $\Delta < 0$, then $f(s_1) > 0$ and if $\Delta > 0$, then either $f(s_1) < 0$ or else q is positive.

On the other hand,

$$f(s_2) = \left(\sqrt{\frac{-p}{3}}\right)^3 + p\left(\sqrt{\frac{-p}{3}}\right) + q = \frac{2p}{3}\sqrt{\frac{-p}{3}} + q$$

This is negative exactly when

$$q < \frac{-2p}{3}\sqrt{\frac{-p}{3}}$$

If $q < 0$, then this is automatic. If $q > 0$, then the inequality stays put when we square both sides, and we get:

$$q^2 < \frac{4p^3}{27}$$

which happens exactly when $\Delta < 0$. Thus if $\Delta < 0$, then $f(s_2) < 0$ and if $\Delta > 0$, then either $f(s_2) > 0$ or else q is negative.

Putting these together, we see that if $\Delta < 0$, then $f(s_1) > 0$ and $f(s_2) < 0$ as desired. And if $\Delta > 0$ then **either** $f(s_1) < 0$ or else $f(s_2) > 0$ (because q cannot be both positive and negative!) so that there is only one real root.

Example: (a) $z^3 - 3z + 4$ has only one real root because $\Delta = 3$.

(b) $z^3 - 3z + 1$ has three real roots because $\Delta = -\frac{3}{4}$.

(c) $z^3 - 3z + 2$ has less that three roots because $\Delta = 0$. Since there are less than three, they must all be real (why?)

For your amusement, and just to see that it is possible, here is:

The Quartic Formula: We wish to solve the equation

$$az^4 + bz^3 + cz^2 + dz + e = 0$$

Preliminary Step: Divide through by a. We get:

$$z^4 + \frac{b}{a}z^3 + \frac{c}{a}z^2 + \frac{d}{a}z + \frac{e}{a} = 0$$

First Step: Complete the **quartic**. This gives us:

$$w^4 + pw^2 + qw + r = 0$$

after the substitutions:

$$w = z + \frac{b}{4a}$$

$$p = \frac{c}{a} - \frac{3b^2}{8a^2}$$

$$q = \frac{d}{a} - \frac{bc}{2a^2} + \frac{b^3}{8a^3}$$

$$r = \frac{e}{a} - \frac{bd}{4} + \frac{b^2c}{16a^3} - \frac{3b^4}{256a^4}$$

Second Step: (From Out of the Blue) Solve the cubic:

$$g(u) = u^3 - 2pu^2 + (p^2 - 4r)u + q^2 = 0$$

using Cardano's formula, and let s_1, s_2, s_3 be the roots. Then the roots of the quartic are:

$$r_1 = \frac{\sqrt{-s_1} + \sqrt{-s_2} + \sqrt{-s_3}}{2}$$

$$r_2 = \frac{\sqrt{-s_1} - \sqrt{-s_2} - \sqrt{-s_3}}{2}$$

$$r_3 = -\frac{\sqrt{-s_1} + \sqrt{-s_2} - \sqrt{-s_3}}{2}$$

$$r_4 = -\frac{\sqrt{-s_1} - \sqrt{-s_2} + \sqrt{-s_3}}{2}$$

Exercises:

1. Find the roots of the following quadratic equations and graph them:

(a) $z^2 + z + 2$

(b) $z^2 + z - 2$

(c) $z^2 + iz + 2$

(d) $z^2 + z + 2i$

(e) $z^2 + iz + 2i$

2. For each of the following cubic equations, find **all** of the real roots.

(a) $z^3 + z + 2$

(b) $z^3 + z - 3$

(c) $z^3 - 3z + 1$

(d) $z^3 - 3z + 2$

(e) $z^3 + z^2 + z + 1$

(f) $z^3 + z^2 + z + 2$