Mathematics 4300– Foundations of Algebra – Fall 1999

4. Next Stop on the Tour: The Complex Numbers. Notice that positive real numbers have an important additional property, namely:

Every positive real number has exactly one positive nth root. In other words, if we are given a positive real number r, there is a unique **positive** real number s such that:

$$s^n = r$$

in which case we write:

$$s = \sqrt[n]{r}$$

(positive real numbers also have a negative nth root when n is even!)

Proof: This proof goes exactly as the proof for multiplicative inverses. The function:

$$f(x) = x^n$$

is continuous (in fact differentiable, with derivative $f'(x) = nx^{n-1}$). Also, $f(0) = 0$ and $\lim_{x\to+\infty} f(x) = +\infty$ so the intermediate value theorem tells us that the graph of f must cross the line $y = r$ somewhere, say at the point (s, r). This tells us that $f(s) = r$, or in other words, $s^n = r$. But now a question emerges (which I avoided earlier) namely, why doesn't this graph cross the line $y = r$ **more than once** when x is between 0 and $+\infty$? Well, it doesn't cross more than once because $f(x)$ is an **increasing** function when x is a positive number, which we can detect from the fact that its derivative is nx^{n-1}, which is positive when x is positive.

(Notice that the function $f(x) = x^2$ is not increasing when x is negative, which allows r to have a negative square root, also.)

It seems as though with the real numbers we have reached the ultimate number system. After all, there are no more points left on the number-line! However, the real numbers have one basic flaw which turns out to be serious. Namely, **negative real numbers do not have a square root**.

We remedy this situation in a curious way. Namely, we introduce the "purely imaginary number" i and declare that:

$$i^2 = -1$$

This is quite similar (and similarly "un-natural") to our previous introduction of the negative numbers (such as -1), which we declared to be the additive inverses to the natural numbers. And just as we located -1 by expanding the number-ray to a number-line, we locate i by expanding the number-line to the "number-plane". That is, we think of the number-line as the x-axis of the xy-plane, and we place i at the point: $(0,1)$.

The Complex Numbers: We define the complex numbers by setting:

$$\mathbf{C} = \{\text{points on the number-plane}\}$$

Recall that a point in the plane is given by two real coordinates, (s,t) which we can rewrite using vector addition as $(s,0)+(0,t)$ or just:

$$s + ti$$

which is the familiar expression for a complex number.

Relations: We have lost the relations "$>$" and "$<$". It makes no sense to say $0 < i$ or $i < 0$ because i is neither to the right nor left of 0.

However, addition and multiplication still work, and what is more, they continue to have useful geometric meanings.

Addition: $(s+ti)+(u+vi) = (s+u)+(t+v)i$. This is the same as ordinary "vector addition" of points in the plane:

$$(s,t)+(u,v) = (s+u, t+v)$$

And this is consistent with the addition of real numbers, since:

$$(s,0)+(u,0) = (s+u,0)$$

Addition of complex numbers is associative

$$((s,t)+(u,v))+(x,y) = ((s+u)+x, (t+v)+y))$$

$$(s,t)+((u,v)+(x,y)) = (s+(u+x), t+(v+y))$$

and $(s+u)+x = s+(u+x)$ and $(t+v)+y = t+(v+y)$ because addition of real numbers is associative. Similarly,

Addition of complex numbers is commutative,

$0 = 0 + 0i$ **is the additive identity, and**

$-s - ti$ **is the additive inverse to** $s + ti$**.**

All the properties of addition are the properties of addition of **vectors** in a **vector space**. Recall a vector space has **scalar multiplication**, which takes a real number and multiplies it by a vector (i.e. a complex number). We'll also accept this multiplication, which says that:

$$r(s + ti) = rs + rti \quad \text{(or in vector notation)} \quad r(s, t) = (rs, rt)$$

But we need to know how to multiply two **vectors** together!

Multiplication: $(s + ti)(u + vi) = su - tv + (sv + tu)i$. In vector notation this is:

$$(s, t) \cdot (u, v) = (su - tv, sv + tu)$$

but in fact this rule is **forced** upon us if we want multiplication to obey the associative, commutative and distrubutive laws, as well as the property that $i^2 = -1$. That's because these laws tell us that:

$$(s + ti)(u + vi) = su + svi + tui + tv(i^2) = su - tv + (sv + tu)i$$

Multiplication of complex numbers is commutative

$$(s+ti)(u+vi) = su-tv+(sv+tu)i = us-vt+(ut+vs)i = (u+vi)(s+ti)$$

Multiplication of complex numbers is associative

$$((s+ti)(u+vi))(x+yi) = sux - tvx - svy - tuy + (suy - tvy + svx + tux)i$$

and we get the same formula when the parentheses are switched.

Multiplication of complex numbers distributes with addition

(This is left as an exercise.)

$1 = 1 + 0i$ **is the multiplicative identity**

$$1 \cdot (u + vi) = (1 + 0i)(u + vi) = (u - 0) + (v + 0)i = u + vi$$

by our rule for multiplication.

Finally, we want to produce multipicative inverses to nonzero complex numbers. To do this, we introduce a couple of new properties:

Complex Conjugation: The complex conjugate, denoted by putting a line over the top of a complex number, is obtained by reversing the sign of the coefficient of i. Thus:

$$\overline{s+ti} = s - ti \text{ (or in vector notation) } \overline{(s,t)} = (s,-t)$$

This has a geometric interpretation. Complex conjugation can be thought of as flipping the number-plane along the x-axis. Note that the real numbers are precisely the numbers which are fixed under complex conjugation, and the "purely imaginary" numbers ti are precisely the numbers which are sent to their additive inverses.

Absolute Value: The absolute value of a complex (or real) number is its Euclidean distance from the origin $0 = (0,0)$. That is,

$$|s+ti| = \sqrt{s^2+t^2}$$

It is very useful to notice that:

$$|s+ti|^2 = s^2+t^2 = (s+ti)(s-ti) = (s+ti)\overline{(s+ti)}$$

so the absolute value is linked to the complex conjugate. Both are linked to the multiplicative inverse because whenever $s+ti \neq 0$, we have:

$$(s+ti)\left(\frac{s-ti}{s^2+t^2}\right) = 1$$

so that $s+ti$ **has a multiplicative inverse**, namely:

$$(s+ti)^{-1} = \frac{s-ti}{s^2+t^2} = \frac{s}{s^2+t^2} - \frac{t}{s^2+t^2}i$$

Thus:

C is a field!

because all the properties of a field (listed in §3) are satisfied by **C**.

I promised you a geometric interpretation for multiplication of complex numbers. This is best understood in terms of:

Polar Coordinates: If r is a positive real number (or zero) and θ is any real number, then the "polar coordinates":

$$(r;\theta)$$

locate the unique point in the plane which is at the distance r from the origin, and whose position vector is at the angle θ from the positive x-axis (measured counter-clockwise). I have put a semi-colon between r and θ to distinguish this notation from the vector notation for a point. Also note that: **in this course, all angles are measured in radians!!**

Notice that there is some redundancy in polar coordinates. Precisely:

$$(0;\theta) \text{ locates the origin regardless of } \theta \text{ and}$$

$(r;\theta)$ and $(r;\theta+2\pi a)$ locate the same point for any integer a.

Multiplication in polar coordinates: I claim that in polar coordinates, the multiplication of complex numbers becomes:

$$(r;\theta)\cdot(s;\psi) = (rs;\theta+\psi)$$

which is a wonderfully simple geometric description of multiplication.

To prove that multiplication works out so well, we need a:

Translation from polar to vector coordinates: Suppose $(r;\theta)$ are the polar coordinates of a complex number z. Then in vector coordinates, we get $z = (r\cdot\cos(\theta), r\cdot\sin(\theta))$

Recall that cos and sin are defined to be the x and y-coordinates of the point on the unit circle at an angle of θ with the positive real axis. Thus

$$(1;\theta) = (\cos(\theta),\sin(\theta))$$

by definition. And replacing 1 by r multiplies the coordinates through by r. This can be proved using similar triangles.

In order to multiply two complex numbers in polar coordinates, we must first translate them into rectangular coordinates, do the multiplication, and finally translate back into polar coordinates. Let $(r;\theta)$ and $(s;\psi)$ be our two complex numbers, in polar coordinates. Then:

$$(r;\theta)\cdot(s;\psi) = (r\cdot\cos(\theta), r\cdot\sin(\theta))\cdot(s\cdot\cos(\psi), s\cdot\sin(\psi))$$

$= (r\cdot\cos(\theta)s\cdot\cos(\psi) - r\cdot\sin(\theta)s\cdot\sin(\psi), r\cdot\cos(\theta)s\cdot\sin(\psi) + r\cdot\sin(\theta)s\cdot\cos(\psi))$

$= (rs(\cos(\theta)\cos(\psi) - \sin(\theta)\sin(\psi)), rs(\cos(\theta)\sin(\psi) + \sin(\theta)\cos(\psi)))$

Now we use a the angle addition formulas from trigonometry, which tell us that the last line above is equal to:

$$(rs\cdot\cos(\theta+\psi), rs\cdot\sin(\theta+\psi))$$

and finally we notice that in polar coordinates, this is: $(rs;\theta+\psi)$ which is just what we wanted!

A big payoff: Every complex number except 0 has n different nth roots.

Proof: Let z be a complex number, written in polar coordinates as $(r;\theta)$ Then taking an nth power is easy to do in polar coordinates: $(r;\theta)^n = (r^n;n\theta)$ by our multiplication formula. So $(\sqrt[n]{r};\frac{\theta}{n})$ is an nth root of $(r;\theta)$. But so are:

$$(\sqrt[n]{r};\frac{\theta}{n}+\frac{2\pi}{n}),(\sqrt[n]{r};\frac{\theta}{n}+\frac{4\pi}{n}),...,(\sqrt[n]{r};\frac{\theta}{n}+\frac{(2n-2)\pi}{n})$$

and what's more, they are all **different** because the angles do not differ by an integer multiple of 2π. We have n of these in all, so we have found as many nth roots as we wanted. But why aren't there any more? Well, again by the polar coordinates formula, we see that if $(s;\psi)$ is an nth root of $(r;\theta)$, then it must be the case that $s=\sqrt[n]{r}$, the **unique** positive nth root of r. Now, we've located some angles that work, namely the angles $\frac{\theta}{n},\frac{\theta}{n}+\frac{2\pi}{n},\frac{\theta}{n}+\frac{4\pi}{n},...$ evenly spaced between $\frac{\theta}{n}$ and $\frac{\theta}{n}+2\pi$. What happens if ψ is between two of these angles? Then $n\psi$ is between θ and $\theta+2\pi$ (after subtracting off an appropriate multiple of 2π). But this means that $(s;\psi)^n \neq (r;\theta)$. So we have found the only ones.

Example: In polar coordinates, $8i = (8;\frac{\pi}{2})$, and its cube roots are:

$$(2;\frac{\pi}{6}),(2;\frac{5\pi}{6}),\text{ and }(2;\frac{3\pi}{2})$$

which in ordinary notation for complex numbers are:

$$\sqrt{3}+i,-\sqrt{3}+i\text{ and } -2i$$

A remarkable event has occurred! We introduced i in order to allow ourselves to take a square root of -1, and lo and behold, we come up with a number system, the complex numbers, where we can take **all** roots of **all** numbers, even the new complex numbers themselves! This is indeed a remarkable development.

I can't resist finishing this section with:

How mathematicians think about polar coordinates: Recall that the Taylor series for e^z is given by:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + ...$$

and notice that this expansion still makes sense when z is a **complex** number. Moreover, if we take $z = i\theta$ (you'll see why we use θ in a minute) then:

$$\begin{aligned} e^{i\theta} &= 1 + (i\theta) + \tfrac{(i\theta)^2}{2!} + \tfrac{(i\theta)^3}{3!} + \tfrac{(i\theta)^4}{4!} + ... \\ &= 1 + i\theta - \tfrac{\theta^2}{2!} - i\tfrac{\theta^3}{3!} + \tfrac{\theta^4}{4!} + i\tfrac{\theta^5}{5!} - ... \\ &= (1 - \tfrac{\theta^2}{2!} + \tfrac{\theta^4}{4!} - ...) + i(\theta - \tfrac{\theta^3}{3!} + \tfrac{(\theta)^5}{5!} - ...) \end{aligned}$$

and if you recall the Taylor series for $\cos(\theta)$ and $\sin(\theta)$, you see that we get the formula:

$$e^{i\theta} = \cos(\theta) + \sin(\theta)i$$

which is even simpler in polar coordinates:

$$e^{i\theta} = (1; \theta).$$

We can multiply through by a positive real number r, to get:

$$re^{i\theta} = (r; \theta)$$

This gives a new proof for the multiplication formula in polar coordinates:

$$(r; \theta) \cdot (s; \psi) = (re^{i\theta}) \cdot (se^{i\psi}) = rse^{i\theta}e^{i\psi} = rse^{i\theta + i\psi} = rse^{i(\theta+\psi)} = (rs; \theta + \psi)$$

so the angles add because $e^x e^y = e^{x+y}$(!)

Since $(1; \pi) = -1$, this interpretation for polar coordinates gives:

$$e^{i\pi} = -1$$

which is an entirely unexpected link between the numbers: i, π and e.

Exercises:

4.1 Prove that multiplication of complex numbers distributes with addition.

4.2 From the formulas for the complex conjugation and multiplicative inverses, prove the following:

(a) $|\overline{s+ti}| = |s+ti|$

(b) $\overline{(\overline{s+ti})} = s+ti.$

(c) $\overline{s+ti} + \overline{u+vi} = \overline{(s+ti)+(u+vi)}.$

(d) $\overline{s+ti} \cdot \overline{u+vi} = \overline{(s+ti)\cdot(u+vi)}.$

(e) $\overline{(s+ti)^{-1}} = (\overline{s+ti})^{-1}$

4.3 Suppose $z = (r;\theta)$ in polar coordinates. Find the polar coordinates expressions for the following:

(a) $\overline{(r;\theta)}$

(b) $(r;\theta)^{-1}$

4.4 (a) Graph the points $1+i, (1+i)^2, (1+i)^3, ...$ (as far out as you can) on the number-plane. What does this look like?

(b) Suppose $z = s+ti$ in vector coordinates. Then what is z in polar coordinates? (Hint: $r = \sqrt{s^2+t^2}$. But what is θ?)

(c) Use (b) (and a calculator with trig functions) to calculate:

$$(1+2i)^{11}$$

by converting into polar coordinates, taking the power, and converting back. Notice that the answer is of the form $a+bi$ (for a and b integers) but the fastest way to get it involves messy calculations on the calculator!

4.5. List (in vector coordinates) and graph all the 8th roots of 1, and determine which two of them are square roots of 1, of i, of -1 and of $-i$.

Food for thought: We've defined e^z, where z is any complex number. Use this to define s^z where s is any real number and z is any complex number. What is s^i in polar coordinates? And what about a log function? Is there a well-defined function $\log(z)$ inverse to e^z?