New Stabilities for Graded Modules

I-70 Algebraic Geometry

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Hilbert Polynomials

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$, let

$$M_\bullet = \bigoplus \Gamma(\mathbb{P}^n, \mathcal{F}(d))$$

be the associated module over $S = \mathbb{C}[x_0, \ldots, x_n]$ and let:

$$\chi_{\mathcal{F}}(d) = \chi(\mathbb{P}^n, \mathcal{F}(d))$$

be the Hilbert polynomial of $\mathcal{F}$. This is **discrete** invariant: constant on flat families over a connected base. Moreover:

$$\deg(\chi_{\mathcal{F}}) \text{ is the dimension of the support of } \mathcal{F}$$

and $\mathcal{F}$ **has pure dimension** m if $\deg(\chi_{\mathcal{E}}) = m$ for all $\mathcal{E} \subseteq \mathcal{F}$.

Gieseker Slope

χ is computed by the Hirzebruch-Riemann-Roch Theorem:

$$\chi_{\mathcal{F}}(t) = \deg(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{td}(\mathbb{P}^n) \cdot e^{tH})$$

where $\mathrm{ch}(\mathcal{F}), \mathrm{td}(\mathbb{P}^n)$ and H are cohomology classes on $\mathbb{P}^n$. Thus:

$$\chi_{\mathcal{F}}(t) = \mathrm{rk}(\mathcal{F}) \cdot \frac{t^n}{n!} + \text{ lower order}$$

The **Gieseker slope** of $\mathcal{F}$ is:

$$\mu_{\mathcal{F}}(t) = \frac{\chi_{\mathcal{F}}(t)}{\text{leading coefficient}}$$

Gieseker Stability

Definition. (a) $\mathcal{F}$ is **Gieseker/Simpson stable** if:

(i) $\mathcal{F}$ is pure-dimensional and (ii) For all proper subsheaves $\mathcal{E} \subset \mathcal{F}$,

$$\mu_{\mathcal{E}}(t) < \mu_{\mathcal{F}}(t) \text{ as polynomials in } t$$

(b) $\mathcal{F}$ is **semi-stable** if pure-dimensional and there is a filtration:

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_N = \mathcal{F}$$

with each $\mathcal{F}_{i+1}/\mathcal{F}_i$ stable of the same slope.

Remark. (Semi)-stability are open conditions on flat families.

Theorem (Gieseker/Simpson). For fixed Hilbert polynomial χ, there is a projective moduli space $\mathcal{M}_{\mathbb{P}^n}(\chi)$ parametrizing equivalence classes of semi-stable sheaves of Hilbert polynomial χ.

Examples.

n =1. The only stable sheaves are line bundles $\mathcal{O}_{\mathbb{P}^1}(d)$ and skyscraper sheaves $\mathbb{C}_p$.

Remark. Minimal rank pure-dimensional sheaves are stable.

n = 2 Any Hilbert scheme of ideal sheaves.

Note. Stable points of the moduli spaces $\mathcal{M}_{\mathbb{P}^2}(\chi)$ are smooth. This is because stable sheaves are simple, and:

$$\mathrm{Hom}(\mathcal{F}, \mathcal{F}) = \mathbb{C} \cdot \mathrm{id}$$

and Serre duality give the vanishing of obstruction spaces.

n $\geq$ **3** "Pathological" moduli spaces abound. (Murphy's Law).

Theorem

The Gieseker slope

$$\mu_{\mathcal{F}}(t) = \frac{\chi_{\mathcal{F}}(t))}{\text{leading coefficient}}$$

is **not** a good slope when *evaluated* at any t. However:

Theorem. (Altavilla, B, Mu, Petkovic) The rational function:

$$\nu_{\mathcal{F}}(t) = \frac{\chi_{\mathcal{F}}(t)}{\chi'_{\mathcal{F}}(t)}$$

defines a one-parameter family of Bridgeland stability conditions on the derived category $\mathcal{D}^b(\mathbb{P}^n)$ of coherent sheaves on $\mathbb{P}^n$.

In particular, $\nu(t)$ defines a good slope with quasi-projective moduli for coherent sheaves of Castelnuovo-Mumford regularity

$$k = \lceil t \rceil$$

Regularity

Definition. $\mathcal{F}$ is k-regular if:

$$H^i(\mathbb{P}^n, \mathcal{F}(k-i)) = 0 \text{ for all } i > 0$$

Basic Properties. (i) If $\mathcal{F}$ is k-regular, then it is $k+1$-regular.

(ii) $\mathcal{F}$ is k-regular if and only if $\mathcal{F}(k)$ is generated by global sections with linear syzygies, i.e. $\mathcal{F}$ has a free resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-k-n)^{a_n} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-k)^{a_0} \to \mathcal{F} \to 0$$

or, equivalently,

$$\mathcal{F} = [\mathcal{O}_{\mathbb{P}^n}(-k-n)^{a_n} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-k)^{a_0}] \in \mathcal{D}^b(\mathbb{P}^n)$$

Example

Consider the example of the ideal sheaf of three points $Z \subset \mathbb{P}^2$.

(a) If the points are **not** collinear, then I_Z is 2-regular, and:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3)^2 \to \mathcal{O}_{\mathbb{P}^2}(-2)^3 \to I_Z \to 0$$

(b) If the points are collinear, then I_Z is not 2-regular and:

$$\mathcal{O}_{\mathbb{P}^2}(-4) \to \begin{matrix} \mathcal{O}_{\mathbb{P}^2}(-3)^3 \\ \bigoplus \\ \mathcal{O}_{\mathbb{P}^2}(-4) \end{matrix} \to \begin{matrix} \mathcal{O}_{\mathbb{P}^2}(-2)^3 \\ \bigoplus \\ \mathcal{O}_{\mathbb{P}^2}(-3) \end{matrix}$$

is the resolution. But both are 3-regular with resolution:

$$\mathcal{O}_{\mathbb{P}^2}(-5)^3 \to \mathcal{O}_{\mathbb{P}^2}(-4)^9 \to \mathcal{O}_{\mathbb{P}^2}(-3)^7$$

Stabilities on Complexes

The following theorem was a precursor to stability conditions:

Theorem (King '91) Let

$$\mathcal{A}_k = \{F^\bullet = [\mathcal{O}_{\mathbb{P}^n}(-k-n)^{a_n} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-k)^{a_0}]\}$$

be the (abelian) category of complexes. Then any assignment:

$$z_i = z(\mathcal{O}_{\mathbb{P}^n}(-k-i)[i]) \in \mathbb{H}$$

defines a GIT quotient space for the action of $G = \prod GL(a_i)$ on complexes with dimension vector $\underline{a} = (a_n, ..., a_0)$ in which:

$$F^\bullet \text{ has a GIT-stable orbit iff } \arg(\sum z_i b_i) < \arg(\sum z_i a_i)$$

for each dimension vector $\underline{b}$ of a subcomplex $E^\bullet \subset F^\bullet$.

3 Points in $\mathbb{P}^2$

Consider the resolution of 3 non-collinear points in $\mathbb{P}^2$

$$\mathcal{O}_{\mathbb{P}^3}(-3)^2 \to \mathcal{O}_{\mathbb{P}^3}(-2)^3 \in \mathcal{A}_2$$

According to the Theorem of King, we assign two complex vectors:

$$z_1 = z(\mathcal{O}_{\mathbb{P}^3}(-3)[1]) \text{ and } z_0 = z(\mathcal{O}_{\mathbb{P}^3}(-2)) \in \mathbb{H}$$

then we have a GIT quotient of the space of complexes. We need: $\arg(z_1) > \arg(z_0)$, and then $F^\bullet$ is stable if it has no subcomplexes with any of the dimension vectors: $(1,0),(2,0),(1,1),(2,1),(2,2)$

Ideal sheaves are then stable, as are the sheaves:

$$\epsilon : 0 \to \mathcal{O}_l(-3) \to \mathcal{F} \to \mathcal{O}_{\mathbb{P}^2}(-1) \to 0$$

for lines $l \subset \mathbb{P}^2$ and non-trivial extension class.

Our Theorem

The more precise version of our Theorem is:

Theorem. The assignment:

$$z(F^\bullet) = \chi'_{F^\bullet}(t) + i\chi_{F^\bullet}(t) \in \mathbb{C}$$

on complexes maps objects of $\mathcal{A}_{\lceil t \rceil}$ to $\mathbb{H}$.

Proof. Since $\chi(\mathcal{O}(t)) = (t+1)\cdots(t+n)/n!$,

$$\chi(\mathcal{O}) = 1 \text{ and } \chi'(\mathcal{O}) > 0$$

$$\chi(\mathcal{O}(-i)[i]) = 0 \text{ and } \chi'(\mathcal{O}(-i)[i]) < 0$$

for all $i = 1, ..., n$. Moreover, as $t \downarrow -1$, the values:

$$\chi(\mathcal{O}(-i)[i]) \text{ move clockwise, staying within } \mathbb{H}$$

3 Collinear Points in $\mathbb{P}^2$

For $t \in (1, 2]$ the stable objects in $\mathcal{A}_2$ with class $\chi = \chi_{I_Z}(t)$ are:

(i) Ideal sheaves of 3 non-collinear points and

(ii) Sheaves $\mathcal{F}$ from the earlier slide

Where are the ideal sheaves of collinear points?
They are 3-regular, and the sheaf inclusion:

$$\mathcal{O}_{\mathbb{P}^2}(-1) \subset I_Z$$

is an inclusion of complexes that is destabilizing when

$$\nu_{\mathcal{O}_{\mathbb{P}^2}(-1)}(t) \geq \nu_{I_Z}(t)$$

i.e. when $t \leq 2 + \sqrt{6}$. After that, it no longer destabilizes!

Twisted Cubics

The Hilbert scheme of twisted cubic curves contains:

I_C the ideal sheaf of a twisted cubic

$I_{E\cup p}$ plane cubic and general point

$I_{E\cup p^*}$ point in the same plane as E

The latter two have subsheaves:

$$I_p(-1) \subset I_{E\cup p} \text{ and } \mathcal{O}_{\mathbb{P}^3}(-1) \subset I_{E\cup p^*}$$

that destabilize the respective sheaves up until:

$$t \approx 6.24 \text{ and } t \approx 7.47, \text{ respectively}$$

Schmidt and Xia have shown that these are the only "walls" for χ (at which the stable objects change) and they used this to recover the description of the Hilbert scheme due to Kleiman and Piene.

Questions about $\mathbb{P}^n$

We can easily show that if $F^\bullet \in \mathcal{D}^b(\mathbb{P}^n)$, then:

(i) If $F^\bullet$ is not a sheaf, then $F^\bullet \notin \mathcal{A}_k$ for large k, so in particular, $F^\bullet$ is not stable for large t.

(ii) If $\mathcal{F}$ is not pure-dimensional, then it is unstable for large t.

(iii) If $\mathcal{F}$ is Gieseker unstable, then it is unstable for large t.

Question. If $\mathcal{F}$ is Gieseker stable, then is it stable for large t? (True for $n = 2, 3$.)

Difficulty. It is hard to “see” the subcomplexes of a complex!

General Question

Are there other varieties X for which:

$$z = \chi'(t) + i\chi(t)$$

define stability conditions on $\mathcal{D}^b(X)$?
And if so, what are the analogues of the categories $\mathcal{A}_k$?

Examples. All Riemann surfaces ("rotated" standard stability).

All algebraic surfaces of positive signature $K_S^2 > 8\chi(\mathcal{O}_S)$.

Odd dimensional quadrics (with varying exceptional collections).

For surfaces of signature zero, we can get "close:"

$$z_\epsilon(t) = \chi'(t) + i\chi(t) - \epsilon\chi''(t)$$

define stability conditions for $0 < \epsilon << 1$ and it seems to be an interesting question to study the analogues of $\mathcal{A}_k$ for, e.g. Hirzebruch surfaces.