Lecture 6. Manifolds

PCMI Summer 2015 Undergraduate Lectures on Flag Varieties

Lecture 6. Topology, and topological manifolds, with applications to Lie groups and Grasssmannians as extended examples.

Definition 6.1. (a) A *topological space* is a set X equipped with a collection of *open subsets* $U \subset X$ with the following properties:

(i) The empty set and X are open sets.

(ii) The intersection of a finite number of open sets is an open set.

(ii) The union of arbitrarily many open sets is an open set.

(b) The complement of an open set is called a *closed* set.

Example 6.1. In the *Euclidean topology* on $\mathbb{R}^n$, the balls:

$$B_r(p) = \{q \in \mathbb{R}^n \mid |q - p| < r\} \text{ for all } r > 0, p \in \mathbb{R}^n$$

of radius r and center p are open, and every open set is a union of balls.

Verification. To see that this defines a topology on $\mathbb{R}^n$, one needs to check that a finite intersection of unions of balls is a union of balls. This isn't very difficult. It follows from the fact that the intersection of two balls is a (usually infinite) union of balls.

Definition 6.2 A collection of open sets $\{U_\lambda \mid \lambda \in \Lambda\}$ is a *basis* for a topology on X if every open set $U \subset X$ in the topology is a union of (usually infinitely many) open sets from the collection.

Remark. A topology is uniquely determined by any basis of open sets.

Thus, by definition, the balls are a basis for the Euclidean topology, but the collection of balls with *rational* radius and *rational* center are a **countable** basis for the topology.

Definition 6.3. A map $f : X \to Y$ between topological spaces is *continuous* if the inverse image of every open set $U \subset Y$ is open in X.

Exercise 6.1. (a) The identity map id_X is continuous.

(b) A composition of continuous maps is continuous.

(c) The inverse of a continuous map **may not be** continuous!

Moment of Zen. The category $\mathcal{T}op$ of topological spaces is:

(a) The collection of topological spaces, with (b) All continuous maps.

An isomorphism in $\mathcal{T}op$ is called a *homeomorphism*.

Example 6.2. A *polynomial* $P(x_1, ..., x_n)$ is a finite linear combination of monomials (over any field k):

$$P(x_1, ..., x_n) = \sum_{D=(d_1,...,d_n)} c_D x_1^{d_1} \cdots x_n^{d_n}; \quad c_D \in k, d_i \geq 0$$

which naturally defines a *function* from k^n to k. Polynomials over $\mathbb{R}$ (or $\mathbb{C}$) are continuous functions for the Euclidean topology.

Definition 6.4. Any subset $Y \subset X$ of a topological space has an *induced topology*, in which the open subsets of Y are the intersection $U \cap Y$ with open subsets $U \subset X$. The induced topology is rigged so that the inclusion map $i : Y \subset X$ is continuous.

Example 6.3. All balls in $\mathbb{R}^n$ (induced topology) are homeomorphic to each other (easy), including the ball $B_\infty = \mathbb{R}^n$ of infinite radius, but a ball in $\mathbb{R}^n$ is **not** homeomorphic to a ball in $\mathbb{R}^m$ if $n \neq m$ (harder).

The category of topological spaces includes a lot of "pathological" examples, including discrete topologies on arbitrary sets, in which the points are all open, and trivial topologies, in which **only** the empty set and X are open. To get a topology that is closer to geometry, consider:

Definition 6.5. X is *Hausdorff* if each pair of points $p_1, p_2 \in X$ may be "separated" by open sets $U_1, U_2 \subset X$, in the sense that:

$$p_1 \in U_1,\ p_2 \in U_2 \text{ and } U_1 \cap U_2 = \emptyset$$

Example 6.4. (a) The Euclidean topology on $\mathbb{R}^n$ is Hausdorff.

(b) The induced topology on any subset of a Hausdorff topological space is Hausdorff.

Definition 6.6. The topology on a Cartesian product $X_1 \times ... \times X_n$ of topological spaces with a basis consisting of products $U_1 \times ... \times U_n$ of open sets $U_i \subset X_i$ is called the *product topology*.

Exercise 6.2. (a) The Euclidean topology on $\mathbb{R}^n$ coincides with the product topology on $\mathbb{R}^n = \mathbb{R}^1 \times .... \times \mathbb{R}^1$ of the Euclidean topologies on the real line. The former yields a basis of balls, and the latter yields a basis of "boxes" (products of open intervals).

(b) A topological space X is Hausdorff if and only if the diagonal:

$$\Delta := \{(x, x) \mid x \in X\} \subset X \times X$$

is a closed subset of the product (with the product topology).

Definition 6.7. A *topological (continuous) manifold* of dimension n is a Hausdorff topological space M that has a countable basis of open sets, each of which is homeomorphic to a ball in $\mathbb{R}^n$.

What is this saying? First, since every point of M is in an open set homeomorphic to a ball, it says that a point of the manifold cannot distinguish its local environment from that of a ball in $\mathbb{R}^n$. Second, the existence of a countable basis ensures that M is not "globally too big," (e.g. M is not a disjoint union of balls, one for each point of the real line). Finally, the Hausdorff condition ensures, for example, that if a sequence of points of M has a limit, then that limit is unique. A point cannot get arbitrarily close to another point without eventually reaching it on a Hausdorff manifold (see $\mathbb{R}^n$ with the doubled origin).

Example 6.5. Every open subset of a manifold is a *submanifold*.

Example 6.6. Some closed sets in $\mathbb{R}^n$ are manifolds (see Lecture 7), but some are not. A pair of intersecting lines in $\mathbb{R}^2$ is not a manifold, because the point of intersection knows that it is not on an interval, no matter how myopic it is.

An important example is obtained by gluing. The idea is that if a manifold M is a union of finitely many overlapping open submanifolds $U_1, ..., U_m \subset M$, then the double and triple intersections $U_i \cap U_j$ and $U_i \cap U_j \cap U_k$ are enough information to reassemble M from the U_i.

Example 6.7 (Gluing). Gluing data for a finite set $M_1, ..., M_n$ of manifolds (of the same dimension) consists of open sets $U_{i,j} \subset M_i$ and gluing homeomorphisms $f_{i,j} : U_{i,j} \to U_{j,i}$ for each pair i, j satisfying:

(i) $f_{i,j} = f_{j,i}^{-1}$

(ii) For every triple i, j, k,

$$f_{i,j}(U_{i,j} \cap U_{i,k}) = U_{j,i} \cap U_{j,k} \text{ and } f_{j,k} \circ f_{i,j} = f_{i,k}$$

as maps from $U_{i,j} \cap U_{i,k}$ to $U_{k,i} \cap U_{k,j}$

Then the following topological space is a manifold *if it is Hausdorff:*

$$M = \coprod_{i=1}^{n} M_i / (U_{i,j} \sim U_{j,i} \text{ via } f_{i,j})$$

Remark. The notation is that of an *equivalence relation.* Two points:

$$x, y \in M_1 \sqcup M_2 \sqcup ... \sqcup M_n$$

are to be identified if $y = f_{i,j}(x)$ for some pair i, j. The data (ii) says that this is an equivalence relation. In particular, the maps $M_i \to M$ are injective, and the open sets of each of the $M_i \subset M$ are a basis for the topology on M. The key observation is that this topology on M is well-defined because a subset $U \subset M_i \cap M_j$ is open whether U is regarded as a subset of M_i or of M_j because f_{ij} is a *homeomorphism.*

Subexample. Glue two copies of $\mathbb{R}^n$ together along $U_{i,j} = \mathbb{R}^n - \{\vec{0}\}$:

(i) Using the identity homeomorphisms: $f_{i,j} = \mathrm{id}$.

(ii) Using the "length inverting" (self-inverse) homeomorphisms:

$$f_{i,j}(\vec{v}) = \vec{v}/|\vec{v}|^2$$

The first gluing gives $\mathbb{R}^n$ with the "doubled origin," i.e. two points, coming from the origins in each copy of $\mathbb{R}^n$, that cannot be separated by open sets in the glued space. It is therefore not Hausdorff.

The second gluing produces a manifold. The glued manifold may be viewed as $\mathbb{R}^n \cup \{\infty\}$ with additional *infinite* open balls:

$$\{q \in \mathbb{R}^n \mid |q| > r\} \cup \{\infty\}$$

This is homeomorphic to the sphere S^n.

Definition 6.8. (a) A topological space X is *disconnected* if there is a pair of disjoint nonempty open subsets $U_1, U_2 \subset X$ with $U_1 \sqcup U_2 = X$. X is *connected* if it is not disconnected.

(b) A topological space X is *compact* if it is Hausdorff and every cover of X by open sets admits a finite sub-cover.

Example 6.8. (a) $\mathbb{R}^n$ is connected but not compact.

(b) The sphere $M = \mathbb{R}^n \cup \{\infty\}$ is connected and compact.

Exercise 6.3. If X is compact and $f : X \to Y$ is continuous, then:

(a) Every closed subset $Z \subset X$ is compact.

(b) $f(X)$ is compact (with the induced topology from Y).

(c) if Y is Hausdorff, then $f(Z) \subset Y$ is closed for every closed $Z \subset X$.

Remark. If M is a compact connected manifold, then the image of M in any gluing of M with other manifolds is both open and closed, hence a compact M cannot be enlarged by gluing while remaining connected.

Definition 6.9. A subset $X \subset \mathbb{R}^n$ is *bounded* if it is contained in a ball of finite radius (or box of finite side lengths).

Heine-Borel Theorem. A subset of $\mathbb{R}^n$ with the induced topology is compact if and only if it is closed and bounded.

Let's apply this to some of our groups:

Proposition 6.1. Of the groups from Lecture 5 (over $\mathbb{R}$ and $\mathbb{C}$):

(a) $\mathrm{SO}(n, \mathbb{R})$, $U(n)$ and $\mathrm{SU}(n)$ are compact and connected.

(b) $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$ are connected but not compact.

Some Proof. The columns of the groups in (a) are orthogonal unit vectors. This implies that each of the groups is contained in a **sphere** (in the respective spaces $\mathbb{R}^{n^2}$ and $\mathbb{C}^{n^2}$). So these groups are bounded. They are closed because each is the zero locus of a polynomial equation **in the real variables**. It is important to realize that the equation $z\overline{z} = 1$ is a polynomial equation $a^2 + b^2 = 1$ on the real and imaginary parts of $z = a + ib$, but is **not** a polynomial in the variable z! So these groups are compact, by the Heine-Borel theorem.

On the other hand, the one-parameter subgroups (and tori) in the groups in (b) proved that they are **unbounded**, although $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{C})$ are zeroes of the single polynomial equation $\det(A) = 1$ in the real **and complex** variables, respectively. It is a general feature of zero sets of (systems) of polynomials in complex variables, that unless they consist of finitely many points, they are always unbounded. Thus, for example, $\mathrm{SO}(n, \mathbb{C})$ is also unbounded (which can alternately be seen by finding a one-parameter subgroup).

Connectedness of the groups in (a) can be proved by induction by applying the following lemma to their realization as a tower of spheres.

Lemma 6.1. If X is compact and Y is Hausdorff, and $f : X \to Y$ is a surjective, continuous map with connected fibers $f^{-1}(p)$, then:
X is connected if and only if Y is connected.

Proof. If $Y = U_1 \sqcup U_2$ disconnects Y, then $X = f^{-1}(U_1) \sqcup f^{-1}(U_2)$ disconnects X (this only required that f be continuous and surjective). Conversely, if $X = U_1 \sqcup U_2$ disconnects X, then for each $p \in Y$, either $f^{-1}(p) \subset U_1$ or $f^{-1}(p) \subset U_2$ because $f^{-1}(p)$ is connected. Therefore $Y = f(U_1) \sqcup f(U_2)$ disconnects Y, since $f(U_i)$ are closed. □

Remark. (i) This lemma does its job but misses the point, which is that the maps $X_{i+1} \to X_i$ in the towers of spheres are each "bundles" of spheres, which are far better behaved than arbitrary continuous maps. In particular, the compactness is a red herring. We leave the connectedness of the groups in (b) as an ambitious exercise, requiring more topology than I have been willing to put into this lecture. You might give some thought to the semi-simple elements of $\mathrm{SL}(n, \mathbb{C})$, or the inclusions $\mathrm{SU}(n, \mathbb{C}) \subset \mathrm{SL}(n, \mathbb{C}) \subset \mathrm{GL}(n, \mathbb{C}) \subset \mathbb{C}^{n^2}$.

(ii) $O(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{R})$ are not connected because they map continuously onto the disconnected sets $\{\pm 1\}$ and $\mathbb{R}^*$, respectively! In fact, they have two "connected components."

Now let's turn our attention back to the flag varieties:

Projective Space. We'll first describe this over any field, and then focus on the topological properties over $\mathbb{R}$ or $\mathbb{C}$.

Definition 6.10. $\mathbb{P}^n_k$ is the locus of lines through the origin in k^{n+1}.

The *projective coordinates* of a line through the origin are **ratios**:

$$(x_0 : ... : x_n)$$

which represent the equivalence classes of points on the line spanned by $(x_0, ..., x_n)$ (and the origin). Evidently, the only restriction on the coordinates is that **some** $x_i \neq 0$.

The set of lines for which a **given** x_i is nonzero may be put in bijection with the points of the "affine hyperplane" $x_i = 1$ via:

$$(x_0 : \cdots : x_i : \cdots x_n) \leftrightarrow \left(\frac{x_0}{x_i}, \cdots, 1, \cdots, \frac{x_n}{x_i}\right)$$

(notice that the colons are replaced with "ordinary" commas).

This **covers** projective space with the affine spaces:

$$U_i = \{(x_{0|i}, \cdots, x_{n|i}) | x_{i|i} = 1\}$$

in coordinates whose indexing reflects the ratios x_j/x_i of the projective coordinates. We now apply Example 6.7 with the gluing data:

$$U_{ij} = \{x \in U_i \mid x_{j|i} \neq 0\}, \quad f_{ij}(x_{0|i}, ..., x_{n|i}) = (x_{0|i}/x_{j|i}, ..., x_{n|i}/x_{j|i})$$

and amusingly, the gluing criteria **has** to apply, because we know, a priori, that gluing is an equivalence relation since we are gluing together subsets of an **existing** locus (of lines). But just to make sure:

$$f_{ji} \circ f_{ij}(\dots x_{k|i} \dots) = f_{ji}(\dots x_{k|i}/x_{j|i} \dots) = (\dots x_{k|i} \cdot x_{j|i}/x_{j|i} \cdot x_{i|i} \dots)$$

the point being that $x_{i|i} = 1$. And then additionally,

$$f_{jk} \circ f_{ij}(\dots x_{l|i} \dots) = f_{jk}(\dots x_{l|i}/x_{j|i} \dots) = (\dots (x_{l|i}/x_{j|i})(x_{j|i}/x_{k|i}) \dots)$$

$$= (\dots x_{l|i}/x_{k|i} \dots) = f_{ik}(\dots x_{l|i} \dots)$$

There are two important things to notice here:

(i) The open sets $U_{ij} \subset U_i$ are each the complement of a hyperplane.

(ii) The gluing maps are rational maps in the coordinates.

Exercise 6.4. (a) When endowed with the topology from gluing, projective space over $\mathbb{R}$ or $\mathbb{C}$ is Hausdorff, and a manifold.

(b) Each projective space over $\mathbb{R}$ or $\mathbb{C}$ is the image of a continuous map from a sphere. Conclude that it is compact and connected.

This is the covering of projective space by the open sets U_i that was promised in the introduction (for Grassmannians and flag manifolds). Next, we turn to the stratification:

Definition 6.11. A *partial ordering* is a binary relation $\preceq$ on a set P that is reflexive, anti-symmetric and transitive, i.e.

- $\mu \preceq \mu$ for all $\mu \in P$
- If $\mu \preceq \nu$ and $\nu \preceq \mu$, then $\mu = \nu$
- If $\mu \preceq \nu$ and $\nu \preceq \lambda$, then $\mu \preceq \lambda$

It is a *total ordering* if $\mu \preceq \nu$ or $\nu \preceq \mu$ for all μ, ν.

Let X be a topological space.

Definition 6.12. (a) A subset $V \subset X$ is *locally closed* if there are open and closed sets $U, Z \subset X$ such that $V = U \cap Z$.

(b) The *closure* $\overline{Y}$ of a subset $Y \subset X$ is the intersection of all closed sets containing Y. It is the unique smallest closed set containing Y.

(c) A *stratification* of X is a set of locally closed subsets $Y_\mu \subset X$ indexed by a partially ordered set P such that:

(i) $X = \sqcup_{\mu \in P} Y_\mu$ is the disjoint union of the sets V_μ.

(ii) Each closure $\overline{Y}_\mu = \sqcup_{\mu \preceq \nu} Y_\nu$

i.e. the *boundary* $\overline{Y}_\mu - Y_\mu$ is the union of **all** the "strictly larger" strata.

Observation. Projective space is stratified by affine spaces with the totally ordered set $P = \{0, 1, ..., n\}$ as follows:

Step 1. Let $V_i = \langle e_0, .., e_i \rangle$ be the standard flag:

$$V_0 \subset V_1 \subset \cdots \subset V_i \subset V_{i+1} \subset \cdots \subset V_n = k^{n+1}$$

Step 2. Consider the following nested closed subsets of $X = \mathbb{P}^n_k$:

$$Z_i = \{l \in \mathbb{P}^n \mid l \subset V_{n-i}\} = \{(x_0 : \cdots x_n) \mid x_{n-i+1} = \cdots = x_n = 0\}$$

Each of these is closed (and is a smaller projective space) because:

$$Z_i = \mathbb{P}^n - (U_n \cup U_{n-1} \cup \cdots \cup U_{n-i+1}), \text{ and}$$

this defines a *stratification* with strata:

$$Y_i := Z_i - Z_{i+1} = \{(*, *, ..., *, 1, 0, ..., 0)\} = Z_i \cap U_{n-i}$$

Remark. For a line $l \subset V_n$ through the origin and a subspace $W \subset V_n$, either the intersection $l \cap W = 0$, in which case the line and subspace are *transverse*, or else $l \cap W = l$. The locally closed sets Y_i are the sets of lines that fail to be transverse to V_{n-i} but are transverse to V_{n-i-1}.

Grassmannians. We are now in position to make sense of comments in the introduction on Grassmannians. The strategy for constructing Grassmannians follows that of projective space.

Definition 6.13. The *Grassmannian* $\mathbb{G}(m,n)$ over a field k is the locus of m-dimensional subspaces $W \subset k^n$.

Analogous to the projective coordinates are *Grassmann* coordinates:

$$(x_{ij}) = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n-1} & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n-1} & x_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn-1} & x_{mn} \end{bmatrix}$$

that define a point of the Grassmannian (the span of the row vectors) provided that the rows are linearly independent, i.e. provided that $\det(x_J) \neq 0$ for *some* multi index $J = \{1 \leq j_1 < \cdots < j_m \leq n\}$ and square matrix $(x_J) := (x_{ij_k})$. The **ambiguity** in the coordinates, analogous to the "ratio" ambiguity for projective coordinates, is:

$$(Ax_{ij}) \sim (x_{ij})$$

where $A \in \mathrm{GL}(m)$ acts by left multiplication on the Grassmann coordinates (x_{ij}), taking one basis for W (as a subspace of k^n) to another. Notice that $\det(Ax_J) = \det(A)(x_J)$ for each of the square matrices of coordinates, and therefore the "Plücker point"

$$\phi(W) := (\cdots : x_J : \dots) \in \mathbb{P}^{\binom{n}{m}-1}$$

only depends upon W and not on the choice of Grassmann coordinates.

Example 6.9. The Plücker point of the plane in $\mathbb{C}^4$ with coordinates:

$$\begin{bmatrix} x_{11} & x_{12} & 1 & 0 \\ x_{21} & x_{22} & 0 & 1 \end{bmatrix}$$

and (total) ordering of the multi-indices J given by:

$$(\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\})$$

is $(x_{11}x_{22} - x_{12}x_{21}, -x_{21}, x_{11}, -x_{22}, x_{12}, 1)$.

Proposition 6.2. The Plücker map $\phi : \mathbb{G}(m,n) \to \mathbb{P}^{\binom{n}{m}-1}$ is injective.

Proof. Fix $W \in \mathbb{G}(m,n)$. As in Example 6.8, assume that (x_{i,j_k}) is the identity for $J \subset [n]$. This gives Plücker coordinate $x_J = 1$. For each $i = 1, ..., m$ and $j \in J^c$, let $J(i,j) \subset [n]$ be defined by: $J(i,j) = J - \{j_i\} \cup \{j\}$. Then the Grassmann coordinate x_{ij} is equal to either $x_{J(i,j)}$ or $-x_{J(ij)}$. In other words, **every** Grassmann coordinate for W is uniquely recovered from the Plücker coordinates. $\square$

Remark. All the other Plücker coordinates are polynomials in the x_{ij}!

Example 6.9 (cont). In the coordinates:

$$\begin{bmatrix} 1 & y_{11} & 0 & y_{12} \\ 0 & y_{21} & 1 & y_{22} \end{bmatrix}$$

(and the same ordering of subsets), the Plücker coordinate is:

$$(y_{21}, 1, y_{22}, y_{11}, y_{11}y_{22} - y_{12}y_{21}, -y_{12})$$

and the *gluing data* from x to y coordinates is therefore:

$y_{11} = x_{22}/x_{21},\ y_{12} = 1/x_{21},\ y_{21} = (x_{12}x_{21} - x_{11}x_{22})/x_{21},\ y_{22} = -x_{11}/x_{21}$

The **gluing data** in general for the Grassmannian can therefore be phrased in terms of the effect of changing coordinates on its image under the Plücker embedding. As in the case of projective space, we let $x_{ij|J}$ denote the Grassmann i,j-coordinate **in a matrix with the Jth minor set to the identity**. Then:

$$U_J = \{(\dots x_{ij|J} \dots) \mid x_{ij_k|J} = \text{id}\}, \quad U_{JL} = \{x \in U_J \mid x_L \neq 0\}$$

$$f_{JL}(\dots x_{ij|J} \dots) = (\dots \pm\ x_{L(i,j)}/x_L \dots)$$

where $L = (l_1 < ... < l_k < ... < l_m)$ determines the coordinates we are transitioning into, and $L(i,j) = (l(i,j)_1 < ... < l(i,j)_k ... < l(i,j)_m)$ recovers the Grassmann coordinates from the Plücker coordinates in the new system.

Remark. The desired properties of the gluing data follow from the fact that we have identified each U_J with its image under the Plücker embedding in projective space and deduced the gluing from the gluing for open sets of projective space. Also note that the open sets U_J are vector spaces (more precisely, affine spaces), that each $U_{JL} \subset U_J$ is the complement of the locus of zeroes of a polynomial, and that all the gluing is by rational functions in the coordinates.

Now it is time to talk about **Schubert Cells** in the Grassmannian. Fix the standard flag $V_i = \langle e_1, ..., e_i \rangle$:

$$V_1 \subset V_2 \subset \cdots \subset V_n = k^n$$

and let $J_{\max} = (n - m + 1 < ... < n)$ be the largest multi index. Then:

$$W \in U_{J_{\max}} \Leftrightarrow \dim(W \cap V_i) = \max\{0, i - m\}$$

i.e. the dimension sequence is: $0, 0, 0..., 0, 1, 2, 3, ..., m$. To see this, recall that the Grassmann coordinates for $W \in U_{J_{\max}}$ can be normalized so that the last minor is the identity matrix. Then the dimension sequence becomes clear.

Such a subspace W is transverse to each V_i. For an arbitrary W, consider the dimension sequence $d_i := \dim(W \cap V_i)$, which increases by 0 or 1 each time with m of the latter. Record the "jump" subsequence of the m times when the dimension jumps by 1:

$$d_{i_1} < ... < d_{i_m}$$

and subtract these from the "generic" jump sequence:

$$n - m + 1 < n - m + 2 < ... < n - 1 < n$$

to obtain the "deficit jump sequence" $\lambda_j = (n - m + j) - d_{i_j}$ of W.

Remark. The deficit jump sequence is non-increasing!

Definition 6.14. The Schubert cell Y_λ associated to a *Young diagram*:

$$\lambda = \{n - m \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0\}$$

is the set of subspaces $W \subset k^n$ with deficit jump sequence λ. Young diagrams have a natural *partial ordering:*

$$\lambda \preceq \mu \text{ if } \lambda_i \leq \mu_i \text{ for all } i$$

and the *size* of a Young diagram λ is $|\lambda| = \lambda_1 + \cdots + \lambda_m$.

Remark. A Young diagram can be visualized as an inverted stack of boxes contained in an $m \times (n - m)$ rectangle with λ_i boxes in the ith row of the diagram.

The subspaces W with Young diagram λ are those with Grassmann coordinates consisting of a 1 in each $(i, n-m+i-\lambda_i)$ position (precisely λ_i to the left of where it would be for a transverse subspace) and 0's below and to the right of each 1. The free coordinates in a matrix with his configuration of 0's and 1's uniquely determine the subspace. Counting these coordinates, we see that:

$$\dim(Y_\lambda) = m(n - m) = |\lambda|$$

Example 6.10. The two two-dimensional cells for $\mathbb{G}(2, 4)$ are:

$$Y_{(2 \geq 0)} \leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{bmatrix}$$

and

$$Y_{(1 \geq 1)} \leftrightarrow \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{bmatrix}$$

Proposition 6.3. $\{Y_\lambda\}$ stratifies $\mathbb{G}(m, n)$.

Proof. It is clear that $\mathbb{G}(m, n)$ is the disjoint union of the Y_λ. To see that the Y_λ are locally closed, simply notice that they are closed subsets of the corresponding U_λ, obtained by setting some of the coordinates to zero. The last part of the stratification condition:

$$\overline{Y_\lambda} = \sqcup_{\lambda \preceq \mu} Y_\mu$$

is more interesting. It suffices to show that

Exercises.

6.1. Show that:

(a) The identity map id_X on a topological space is continuous.

(b) A composition of continuous maps is continuous.

(c) Find a continuous bijection whose inverse is not continuous.

6.2. (a) Show that the Euclidean topology on $\mathbb{R}^n$ coincides with the product topology on $\mathbb{R}^n = \mathbb{R}^1 \times .... \times \mathbb{R}^1$ of the Euclidean topologies on the real line. The former yields a basis of balls, and the latter yields a basis of "boxes" (products of open intervals).

(b) A topological space X is Hausdorff if and only if the diagonal:

$$\Delta := \{(x,x) \mid x \in X\} \subset X \times X$$

is a closed subset of the product (with the product topology).

6.3. If X is compact and $f : X \to Y$ is continuous, show that:

(a) Every closed subset $Z \subset X$ is compact.

(b) $f(X)$ is compact (with the induced topology from Y).

(c) if Y is Hausdorff, then $f(Z) \subset Y$ is closed for every closed $Z \subset X$.

(Hint: Consider the *graph* $\Gamma = \{(z, f(z))\} \subset Z \times Y$ of $f : Z \to W$. If Y is Hausdorff, then Γ is closed and $f(Z)$ is the image of Γ under the projection map $Z \times Y \to Z$.)

6.4. (a) When endowed with the topology from gluing, projective space over $\mathbb{R}$ or $\mathbb{C}$ is Hausdorff, and a manifold.

(b) Each projective space over $\mathbb{R}$ or $\mathbb{C}$ is the image of a continuous map from a sphere. Conclude that it is compact and connected.

(c) Show that the Grassmannians are compact, connected manifolds.

6.5. Prove a symmetry among the Young diagrams indexing the Schubert cells of $\mathbb{G}(m, n)$. Namely, show that the number of such diagrams of size i is the same as the number of size $m(n-m) - i$.

The cells of a particular length $|\lambda|$ are a basis for the *cohomology* of the complex Grassmannian in codimension λ, so that the number of such cells of length i is a "$2i$th betti number" of the complex Grassmannian. this symmetry is a consequence of Poincaré duality. When you look at the diagrams of complementary sizes, try to find an explicit bijection between them. This is the pairing under Poincaré duality.

6.6. Consider the Grassmannian $\mathbb{G}(m,n)$ over a **finite field** $\mathbb{F}_q$ with q elements. How many points are there in this Grassmannian? What happens when you set $q = 1$? (Try the case of projective space first.)

6.7. Prove that $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{C})$ are connected.

6.8. Discuss the connectedness and/or compactness of the symplectic groups $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{Sp}(2n,\mathbb{C})$.

6.9. Are the groups $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ manifolds? What about the groups $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{C})$? Or the others?