PCMI Summer 2015 Undergraduate Lectures on Flag Varieties

0. What are Flag Varieties and Why should we study them?

Let's start by over-analyzing the binomial theorem:

$$(x+y)^n = \sum_{m=0}^{n} \binom{n}{m} x^m y^{n-m}$$

has *binomial coefficients*

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

that count the number of m-element subsets T of an n-element set S.

Let's count these subsets in two different ways:

(a) The number of ways of **choosing** m different elements from S is:

$$n(n-1)\cdots(n-m+1) = n!/(n-m)!$$

and each such choice produces a subset $T \subset S$, **with an ordering**:

$$T = \{t_1, ..., t_m\} \text{ of the elements of } T$$

This realizes the desired set (of subsets) as the image:

$$f : \{T \subset S, T = \{t_1, ..., t_m\}\} \to \{T \subset S\}$$

of a "forgetful" map. Since each preimage $f^{-1}(T)$ of a subset has $m!$ elements (the orderings of T), we get the binomial count.

Bonus. If $S = [n] = \{1, ..., n\}$ is ordered, then each subset $T \subset [n]$ has a *distinguished ordering* $t_1 < t_2 < ... < t_m$ that produces a "section" of the forgetful map. For example, in the case $n = 4$ and $m = 2$, we get:

$$\{T \subset [4]\} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$$

realized as a subset of the set $\{T \subset [4], \{t_1, t_2\}\}$.

(b) The group $\text{Perm}(S)$ of permutations of S "acts" on $\{T \subset S\}$ via

$$f(T) = \{f(t) \mid t \in T\} \text{ for each permutation } f : S \to S$$

This is an transitive action (every subset maps to every other subset), and the "stabilizer" of a particular subset $T \subset S$ is the subgroup:

$$\text{Perm}(T) \times \text{Perm}(S - T) \subset \text{Perm}(S)$$

of permutations of T and $S - T$ separately. This gives the equality:

$$|\{T \subset S\}| = \frac{|\text{Perm}(S)|}{|\text{Perm}(T)| \cdot |\text{Perm}(S-T)|} = \frac{n!}{m!(n-m)!}$$

Bonus. The "action" $\text{Perm}(S) \to \text{Perm}(\{T \subset S\})$ of permutations on subsets takes permutations of n things to permutations of $\binom{n}{m}$ things.

Now let's replace finite sets with finite-dimensional **vector spaces.** Let V be a fixed vector space of dimension n, and consider:

$$\text{Grass}(m, V) := \{\text{subspaces } W \subset V \mid \dim(W) = m\}$$

This is called the **Grassmannian**, which we can study in two ways, analogous to (a) and (b) above.

(a) The "number of ways" to choose m linearly independent vectors from V is a (Zariski) open subset U of a vector space of dimension mn:

$$U \subset V \times ... \times V = V^m$$

Thus, $U = \{W \subset V, W = \langle \vec{w}_1, ..., \vec{w}_m \rangle\}$ and we have a forgetful map:

$$f : U \to \text{Grass}(m, V) = \{W \subset V\}$$

from a space we understand to the Grassmannian. In this case, $f^{-1}(W)$ is the set of bases for W, which is (a principal homogeneous space for) $\text{GL}(W)$, the group of invertible $m \times m$ matrices. Since $\text{GL}(W) \subset W^m$ is an open subset of a vector space of dimension m^2, we conclude that the dimension of the Grassmannian is $mn - m^2 = m(n - m)$.

Bonus. Suppose $V = \mathbb{R}^n$ (or $\mathbb{C}^n$). Then the vectors of V are n-tuples:

$$\vec{v} = (v_1, ..., v_n)$$

and we can write each choice $\vec{w}_1, ..., \vec{w}_m$ as the rows of an $m \times n$ matrix:

$$\begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,n} \\ w_{2,1} & w_{2,2} & \cdots & w_{2,n} \\ \vdots & \vdots & & \vdots \\ w_{m,1} & w_{m,2} & \cdots & w_{m,n} \end{bmatrix} = \begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \vdots \\ \vec{w}_m \end{bmatrix}$$

and we note that the vectors $\vec{w}_1, ..., \vec{w}_m$ are linearly independent if and only if **some** $m \times m$ *minor* of the $m \times n$ matrix is invertible.

This time there is no section of the forgetful map. In other words, there is no distinguished choice of basis for each $W \subset \mathbb{R}^n$ analogous to the distinguished ordering of the elements of each subset $T \subset [n]$. There are two interesting ways to try to get around this.

We'll illustrate both attempts in the case of $\text{Grass}(2, \mathbb{R}^4)$.

First Attempt. Cover the Grassmannian with "open" vector spaces.

For each subset $T \subset [4]$ consider only the matrices that are invertible with respect to the minor indexed by the elements of T, and normalize these bases with row operations. This gives a distinguished choice of basis for such matrices:

$$\{1,2\} \leftrightarrow \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}, \{1,3\} \leftrightarrow \begin{bmatrix} 1 & * & 0 & * \\ 0 & * & 1 & * \end{bmatrix}$$
$$\{1,4\} \leftrightarrow \begin{bmatrix} 1 & * & * & 0 \\ 0 & * & * & 1 \end{bmatrix}, \{2,3\} \leftrightarrow \begin{bmatrix} * & 1 & 0 & * \\ * & 0 & 1 & * \end{bmatrix}$$
$$\{2,4\} \leftrightarrow \begin{bmatrix} * & 1 & * & 0 \\ * & 0 & * & 1 \end{bmatrix}, \{3,4\} \leftrightarrow \begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix}$$

This produces an **open cover**:

$$\mathrm{Grass}(m, \mathbb{R}^n) = \bigcup_{T \subset [n]} U_T \text{ together with sections } U_T \to U$$

of the forgetful map. The sections do not "fit together" over $U_T \cap U_{T'}$, however, and that failure can be captured in a "cocycle."

Second Attempt. Stratify $\mathrm{Grass}(m, \mathbb{R}^n)$ with "locally closed" spaces.

Use the "natural" *partial ordering* on (ordered!) subsets $T \subset [4]$:

$$\{1,2\} \prec \{1,3\} \prec \{1,4\}, \{2,3\} \prec \{2,4\} \prec \{3,4\}$$

and modify the matrices above by removing any subspaces that were already accounted for by a previous set T. Thus:

$$\{1,2\} \leftrightarrow \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix} = \mathbb{R}^4$$

$$\{1,3\} \leftrightarrow \begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix} = \mathbb{R}^3$$

$$\{1,4\} \leftrightarrow \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbb{R}^2 = \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix} \leftrightarrow \{2,3\}$$

$$\{2,4\} \leftrightarrow \begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbb{R}^1$$

$$\{3,4\} \leftrightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbb{R}^0$$

Whereas the first attempt produced six overlapping open subsets of $\mathrm{Grass}(2, \mathbb{R}^4)$ with sections, this attempt in contrast produces six spaces:

$$\mathbb{R}^4 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^2 \sqcup \mathbb{R}^2 \sqcup \mathbb{R}^1 \sqcup \mathbb{R}^0 = \mathrm{Grass}(2, \mathbb{R}^4)$$

that fit together like the pieces of a jigsaw puzzle, with no overlaps.

(b) The group $\mathrm{GL}(n,\mathbb{R})$ of invertible $n\times n$ matrices acts transitively on each Grassmannian $\mathrm{Grass}(m,\mathbb{R}^n)$ via $A(W) = \{A(\vec{w})\ |\vec{w} \in W\}$. The stabilizer of the "first" subspace $\langle e_1, ..., e_m\rangle$ consists of invertible $n \times n$ matrices with a rectangle of $m(n-m)$ zeroes , and therefore:

$$\mathrm{Grass}(m,\mathbb{R}^n) = \frac{\mathrm{GL}(n,\mathbb{R})}{\mathrm{Stab}(\langle e_1, ..., e_n\rangle)}$$

has dimension $m(n-m)$. E.g. The stabilizer of $\langle e_1, e_2\rangle \in \mathrm{Grass}(2,\mathbb{R}^4)$ is:

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

Bonus. Here the "action":

$$\mathrm{GL}(n,\mathbb{C}) \to \mathrm{Aut}(\mathrm{Grass}(m,\mathbb{C}^n));\ A(\vec{v}) \mapsto A(W)$$

takes automorphisms of $\mathbb{C}^n$ (invertible matrices) to automorphisms of the *projective variety* $\mathrm{Grass}(m,\mathbb{C}^n)$, which can be used in creative ways to construct actions on vector spaces of many different dimensions.

Generalizations. (i) Multinomials and Flag Varieties.

The "multinomial theorem" gives:

$$(x_1 + \cdots + x_k)^n = \sum_{m_1+\cdots+m_k=n} \binom{n}{m_1, ..., m_k} x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}$$

where the *multinomial*

$$\binom{n}{m_1, ..., m_k} = \frac{n!}{m_1! \cdots m_k!}$$

is the size of the set $\mathrm{Fl}(n_1, ..., n_k)(S)$ of **flags** of subsets:

$$T_1 \subset T_2 \subset \cdots \subset T_k = S \text{ with } |T_i| = n_i := m_1 + \cdots + m_i$$

Suppose the elements of one particular such flag are:

$$\{s_1, ..., s_{n_1}\} \subset \{s_1, ..., s_{n_2}\} \subset \cdots \subset \{s_1, \dots, s_n\} = S$$

and the elements of another are:

$$\{t_1, ..., t_{n_1}\} \subset \{t_1, ..., t_{n_2}\} \subset \cdots \subset \{t_1, \dots, t_n\} = S$$

Then the permutation $f : S \to S$ given by $f(s_i) = t_i$ takes the first flag to the second. So $\mathrm{Perm}(S)$ acts transitively on the set of flags, with stabilizer $\mathrm{Perm}(T_1) \times \mathrm{Perm}(T_2 - T_1) \times ... \times \mathrm{Perm}(T_k - T_{k-1})$ giving the count of the multinomial coefficient analogous to the count in (b).

The analogue of the count in (a) is also available, with the set of ordered flags defined exactly as before and the forgetful map:

$$f : \text{OrderedFlags} \to \text{Flags}$$

The **full flags** are a particularly interesting case:

$$\binom{n}{1, ..., 1} = n!$$

in which the elements of the flags are already ordered by the flag, and the forgetful map forgets nothing. Thus, the elements of the full flag set are in a bijection with the $n!$ permutations of $[n]$.

Passing to a vector space V, we wish to analyze the *flag variety*

$$\text{Fl}(n_1, n_2, ..., n_k)(V) = \{W_1 \subset ... \subset W_k = V\}$$

of flags of subspaces of dimension $n_1, ..., n_k = n$. Let's skip to the full flags (the others are a hybrid between full flags and Grassmannians). In this case, approach (b) gives:

$$\text{Fl}(1, 2, 3, ..., n)(\mathbb{C}^n) = \text{GL}(n, \mathbb{C})/\text{Stab}(\mathbb{C}^1 \subset \mathbb{C}^2 \subset ... \subset \mathbb{C}^n)$$

and the stabilizer is the group of (invertible) upper triangular matrices!

Approach (a) (or rather, the Bonus) is also interesting in this case.

First Attempt. Cover the full flag variety with open vector spaces.

In this case, we associate to each permutation σ the "reference flag:"

$$\langle e_{\sigma(1)} \rangle \subset \langle e_{\sigma(1)}, e_{\sigma(2)} \rangle \subset \cdots$$

and cover the full flag variety with the open sets $U_\sigma \subset \text{Fl}(1, 2, ..., n)$ consisting of flags that are "transverse" to the reference flag. In the nearly trivial case of $\text{Fl}(1, 2)$, this would give:

$$U_{\text{id}} = \begin{bmatrix} * & 1 \\ 1 & 0 \end{bmatrix} \text{ and } U_{(1\ 2)} = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$$

Second Attempt. Stratify the full flag variety with vector spaces. This requires **ordering** the permutations and removing redundant flags, which we will do once we have a better grasp of permutations. In our trivial example, we get:

$$\text{id} \leftrightarrow \begin{bmatrix} * & 1 \\ 1 & 0 \end{bmatrix} \text{ and } (1\ 2) \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Incidentally, this trivial example exhibits the complex projective line $\mathbb{CP}^1 = \text{Fl}(1, 2)$ as a union of two copies of the complex line and as one complex line with a point "at infinity", respectively.

(ii) Flags for other Lie Groups.

Consider the complex dot product:

$$\vec{v} \cdot \vec{w} = \sum_{i=1}^{n} v_i w_i$$

(this is not the Hermitian inner product). We may consider varieties of (full) flags of isotropic or coisotropic subspaces, i.e. subspaces $W \subset \mathbb{C}^n$ such that either $W^\perp \subseteq W$ or $W \subseteq W^\perp$ (this is a strange concept, but over the complex numbers the dot product is not positive definite, and vectors can be self-orthogonal!). In this case, the variety:

$$\mathrm{Isot}(1, 2, 3, ..., n)(\mathbb{C}^n)$$

of isotropic flags of $\mathbb{C}^n$ admits a transitive action of the Lie Group $\mathrm{SO}(n, \mathbb{C})$ of "orthogonal" $n \times n$ matrices of determinant 1 that satisfy:

$$AA^T = \mathrm{id}$$

(the parity of n...even or odd...turns out to be important).

Similarly, we can consider the *symplectic* matrices that fix a "skew" inner product on an even dimensional space, acting on a flag variety of isotropic subspaces for a skew inner product. In all cases, the flag varieties may be either covered by open sets indexed by the elements of some "Weyl" group or else stratified by sets ordered by a "Bruhat" ordering on the Weyl group, and the action of the Lie group on the flag variety produces interesting finite-dimensional representations.

My plan for these lectures is as follows:

- Basic set theory (the category of sets).
- Basic linear algebra (the categories of vector spaces).
- Groups and Actions.
- G-modules for finite groups.
- Basic Topology, Orthogonal and Unitary Groups
- Differential Geometry and Lie Algebras
- Projective spaces
- Grassmannians
- Flag Manifolds
- Bruhat Orderings and Schubert Varieties
- Representations of complex Lie Groups

An analogy between finite sets and finite-dimensional vector spaces is a theme in these lectures. This analogy runs pretty deep. Here is a table of some similarities and differences.

Category	Sets	Vector Spaces
Operations	None	+ and Scalar $\cdot$
Minimal object	Empty set	Zero space
Subobjects	Subsets	Subspaces
Morphisms	Maps	Linear Maps
Quotients	None	Spaces of Cosets
Dual	None	Dual Space
Size	Cardinality	Dimension
Standard Objects	$[n] = \{1, 2, ..., n\}$	k^n
Standardizing	Ordering	Choice of Basis
Notation for Maps	Tuples and Cycles	Matrices
Automorphisms	Permutations	Invertible Matrices
Aut(Standard)	Perm(n)	GL(n)
Characteristic Function	Sign	Determinant
Special Subgroups	Alt(n)	SL(n)
Other Subgroups	Finite	Linear
Conjugacy Classes	Partitions	Jordan Blocks
Flags	Multinomial	Flag manifolds

If there were a bumper sticker for this course, it would be:

"Think categorically. Act automorphically."