Lecture 4. G-Modules

PCMI Summer 2015 Undergraduate Lectures on Flag Varieties

Lecture 4. The categories of G-modules, mostly for finite groups, and a recipe for finding "every" irreducible G-module of a finite group G.

Let's start by thinking about direct sums of vector spaces:

Definition 4.1. A vector space V is the *direct sum*:

$$V = U_1 \oplus \cdots \oplus U_m$$

of subspaces $U_1, ..., U_m \subset V$ if each $\vec{v} \in V$ has a **unique** expression: $\vec{v} = \vec{u}_1 + ... + \vec{u}_m$ as a sum of vectors $\vec{u}_i \in U_i$ from each subspace.

Example 4.1. (a) If $\vec{v}_1, ..., \vec{v}_n \in V$ are a basis, then V is the direct sum of the one-dimensional subspaces (i.e. lines) $L_i := \{c\vec{v}_i \mid c \in k\}$.

(b) If $U, W \subset V$ are a pair of subspaces that satisfy:

$$U \cap W = \{\vec{0}\} \text{ and } \dim(U) + \dim(W) = \dim(V),$$

then $V = U \oplus W$. (Exercise 4.1)

(c) Three distinct lines L_1, L_2, L_3 in the xy-plane inside $\mathbb{R}^3$ satisfy:

$$L_i \cap L_j = \{\vec{0}\} \text{ and } \dim(L_1) + \dim(L_2) + \dim(L_3) = 3$$

but $\mathbb{R}^3$ is not a direct sum of L_1, L_2, L_3.

Remark. Since every finite dimensional vector space has a basis, it follows that every vector space is the direct sum of lines. We say that all vector spaces other than lines and $\{\vec{0}\}$ are *reducible*.

Proposition 4.1. Suppose V has finite dimension and $U \subset V$ is a subspace. Then there is a complementary subspace $W \subset V$ such that:

$$q : W \to V/U$$

is an isomorphism, and $V = U \oplus W$.

Proof. Let $\vec{u}_1, ..., \vec{u}_m$ be a basis of U, and add linearly independent vectors (in V) until a basis: $\vec{u}_1, ..., \vec{u}_m, \vec{w}_1, ..., \vec{w}_{n-m}$ of V is obtained. Then $W = \{c_1\vec{w}_1 + \cdots + c_{n-m}\vec{w}_{n-m}\}$ has the desired properties. □

Remark. The analogue of this Proposition is false for abelian groups. Consider, for example, the subgroup of even integers:

$$2\mathbb{Z} \subset \mathbb{Z} \text{ with coset group } \mathbb{Z}/2\mathbb{Z}$$

Evidently there is no subgroup $A \subset \mathbb{Z}$ that maps isomorphically to $\mathbb{Z}/2\mathbb{Z}$, because such a subgroup would produce an integer $a \in \mathbb{Z}$ with the property that $2a = 0$. The question of whether a group of cosets G/H of a normal subgroup "lifts" to a subgroup of G is subtle.

Let G be a group.

Definition 4.2. (a) A *G-module* is a pair consisting of a vector space V (over a field k) **together with** a representation $\rho : G \to \mathrm{GL}(V)$.

(b) A subspace $U \subset V$ fixed by $\rho(g)$ for all $g \in G$ is a *G-submodule*.

(c) A linear map $f : V \to W$ of G-modules is *G-linear* if:

$$f(g\vec{v}) = gf(\vec{v}) \text{ for all } g \in G \text{ and all } \vec{v} \in V$$

Remark. Although a G-module is a pair, it is often just written as V, with ρ understood. Similarly, unless we need to be careful, we'll write $g\vec{v}$ when we mean $\rho(g)(\vec{v})$ (as we did in (c) above).

Exercise 4.2. (a) The identity map on G-modules is G-linear.

(b) A composition of G-linear maps is G-linear.

(c) The inverse of an invertible G-linear map is G-linear.

(d) The kernel and image of a G-linear map are G-submodules.

(e) The space of cosets of a G-linear map is a G-module in such a way that the map $q : V \to V/U$ is G-linear.

Moment of Zen. The category $G\mathcal{M}od$ (fixed G) consists of:

(a) the collection of all G-modules, and (b) all G-linear maps

Example 4.2. (a) The trivial representation ($\rho(g) = \mathrm{id}_V$ for all g) makes any vector space into a *trivial* G-module. Every subspace of a trivial G-module is a G-submodule.

(b) Let $G = \mathrm{Perm}(n)$ and consider the G-module k^n with the natural representation $\tau \cdot e_i = e_{\tau(i)}$. Then the addition of coordinates:

$$a : k^n \to k;\ a(\vec{v}) = v_1 + ... + v_n$$

is a G-linear map to the **trivial** G-module k since:

$$\tau\vec{v} = (v_{\tau^{-1}(1)}, ..., v_{\tau^{-1}(n)}) \text{ and } a(\tau\vec{v}) = \sum_{i=1}^{n} v_{\tau^{-1}(i)} = a(\vec{v}) = \tau a(\vec{v})$$

The G-submodule $\ker(a)$ is the *standard representation* of $\mathrm{Perm}(n)$.

(c) Consider the two-dimensional standard representation of $\mathrm{Perm}(3)$. In terms of the basis $e_1 - e_2, e_2 - e_3$, matrices for the action include:

$$\rho(1\ 2) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \rho(2\ 3) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

The other matrices are left as an exercise.

Definition 4.3. A G-module V with representation ρ is *irreducible* if the only G-submodules of V are $\{\vec{0}\}$ and V itself.

Example 4.3. (a) Any one-dimensional G-module is irreducible. These include the trivial one-dimensional G-modules, but also any *character*:

$$\chi : G \to k^*; \quad \chi(gh) = \chi(g) \cdot \chi(h)$$

E.g. The characters of C_n from Example 3.7(a) are $\chi_m(x) = e^{\frac{2\pi im}{n}}$.

Consider next the action of D_{2n} from Example 3.7(b), consisting of the rotation action of C_n augmented by the reflections

(b) The rotation action of $C_n \subset D_{2n}$ on $\mathbb{R}^2$ is already irreducible when $n > 2$ because no line is left invariant under the rotations. But:

(c) The same rotation action of C_n on $\mathbb{C}^2$ is *reducible*, with two invariant lines, the eigenspaces of x (and all powers of x), spanned by:

$$(1, i) \text{ and } (1, -i), \text{ respectively}$$

(d) Since the reflection given by y *switches* the two invariant lines for x, it follows that $\mathbb{C}^2$ with the action of D_{2n} is irreducible.

Proposition 4.2. The only irreducible G-modules over $\mathbb{C}$ for an **abelian** group G are the one-dimensional characters of G. In other words, any set of commuting matrices has a common eigenvector.

Proof. If V is a vector space over $\mathbb{C}$, then every element of $\mathrm{GL}(V)$ has an eigenvector. Suppose A and B are commuting elements of $\mathrm{GL}(V)$, and $\vec{v}$ is an eigenvector of A with eigenvalue λ. Then:

$$A(B\vec{v}) = B(A\vec{v}) = B(\lambda\vec{v}) = \lambda(B\vec{v})$$

so $B\vec{v}$ is **another** eigenvector of A with eigenvalue λ. In other words, B acts on the λ eigenspace for A, and therefore B has an eigenvector in this eigenspace, and so A and B **share** an eigenvector.

If $A_1, ..., A_n$ are elements of $\mathrm{GL}(V)$ sharing a common eigenvector and A_{n+1} commutes with each of them, then by the same argument, A_{n+1} acts on the common eigenspaces and therefore shares an eigenvector with $A_1, ..., A_n$. This proves that every finite set of commuting matrices shares an eigenvector. If an infinite set of commuting matrices failed to share an eigenvector, then some finite subset would also fail to share an eigenvector (by induction on the dimensions of the shared eigenspaces), so the result applies to infinite abelian groups as well. □

Example 4.4. Consider the two-dimensional representation of the abelian group $\mathbb{C}$ (with addition):

$$\rho(z) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$$

These matrices share a single one-dimensional eigenspace, the x-axis.

Definition 4.4. A G-module V is the *direct sum*:

$$V = U_1 \oplus \cdots \oplus U_m$$

of G-submodules $U_1, ..., U_m \subset V$ if each $\vec{v} \in V$ has a unique expression: $\vec{v} = \vec{u}_1 + ... + \vec{u}_m$ as a sum of vectors $\vec{u}_i \in U_i$ from each submodule.

Remark. This is **identical** to Definition 4.1, but Proposition 4.1 for G-modules is more subtle. The problem is that complementary subspaces to a G-submodule $U \subset V$ are not usually G-submodules. In Example 4.4 above, the y-axis is complementary to the x-axis, but it is not invariant under the action of $\mathbb{C}$, and in fact, in that Example, there is **no** complementary subspace of $\mathbb{C}^2$ that is a $\mathbb{C}$-submodule!

Consider again the *dot product* on $\mathbb{R}^n$ from Lecture 2. This is *bilinear* (linear on each vector) and **positive definite**:

$$\vec{v} \cdot \vec{v} = \sum_{i=1}^{n} v_i^2 > 0 \text{ unless } \vec{v} = \vec{0}$$

Similarly, there is a *standard Hermitian product* on $\mathbb{C}^n$, namely,

$$\langle \vec{v}, \vec{w} \rangle = \sum_{j=1}^{n} v_j \overline{w}_j \in \mathbb{C}$$

that is linear in the first vector and *conjugate linear* in the second, i.e. $\langle \vec{v}_1 + \vec{v}_2, \vec{w} \rangle = \langle \vec{v}_1, \vec{w} \rangle + \langle \vec{v}_2, \vec{w} \rangle$ and $\langle \vec{v}, \vec{w}_1 + \vec{w}_2 \rangle = \langle \vec{v}, \vec{w}_1 \rangle + \langle \vec{v}, \vec{w}_2 \rangle$ and $\langle c\vec{v}, \vec{w} \rangle = c\langle \vec{v}, \vec{w} \rangle$ but $\langle \vec{v}, c\vec{w} \rangle = \overline{c}\langle \vec{v}, \vec{w} \rangle$. The Hermitian product is conjugate symmetric, in the sense that $\langle \vec{v}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{v} \rangle}$.

Remark. We have avoided complex conjugation thus far, but no longer. If $c = a + ib$, then $\overline{c} = a - ib$, and $c\overline{c} = a^2 + b^2 = |c|^2$. It follows that the Hermitian dot product is also *positive definite*, in the sense that:

$$\langle \vec{v}, \vec{v} \rangle = \sum_{i=1}^{n} |v_i|^2 \text{ is real and positive unless } \vec{v} = 0$$

Let $U \subset V$ be a subspace of a vector space V defined over $\mathbb{R}$ or $\mathbb{C}$.

Definition 4.5. The *orthogonal complement* $U^\perp \subset V$ is the set of vectors in V such that $\vec{u} \cdot \vec{v} = 0$ (if over $\mathbb{R}$) or $\langle \vec{u}, \vec{v} \rangle = 0$ (if over $\mathbb{C}$).

Exercise 4.3. When V is a vector space over $\mathbb{R}$ or $\mathbb{C}$, then $U^\perp \subset V$ is a complementary subspace to $U \subset V$ in the sense of Proposition 4.1. In particular,

$$V = U \oplus U^\perp \text{ and } \dim(U) + \dim(U^\perp) = \dim(V)$$

This doesn't (yet) help with our conundrum. If $U \subset V$ is a G-submodule, there is no reason to expect that $U^\perp$ is also a G-submodule. The key is to recognize that there are **many** positive definite products and to find one that is G-invariant.

Definition 4.6. (a) Let V be a vector space over $\mathbb{R}$. A map

$$b : V \times V \to \mathbb{R}$$

is a *positive definite inner product* if it is symmetric and bilinear and

$$b(\vec{v}, \vec{v}) > 0 \text{ for all } \vec{v} \neq \vec{0}$$

(b) Let V be a vector space over $\mathbb{C}$. A map

$$h : V \times V \to \mathbb{C}$$

is a *positive definite Hermitian product* if it is conjugate symmetric, and linear in the first vector (and conjugate linear in the second), and

$$h(\vec{v}, \vec{v}) > 0 \text{ for all } \vec{v} \neq \vec{0}$$

Note. $h(\vec{v}, \vec{v}) = \overline{h(\vec{v}, \vec{v})}$ is automatically a real number.

Matrix Notation. An inner product is represented by the matrix:

$$B = (b(e_i, e_j)) \text{ with } B = B^T \text{ and } b(\vec{v}, \vec{w}) = (\vec{v})^T B \vec{w}$$

where $\vec{v}$ and $\vec{w}$ are regarded as one-column matrices. Notice that the ordinary dot product is represented by the identity matrix.

Similarly, a Hermitian product is represented by the matrix:

$$H = (h(e_i, e_j)) \text{ with } H = \overline{H}^T \text{ and } h(\vec{v}, \vec{w}) = (\vec{v})^T H \overline{\vec{w}}$$

and the standard Hermitian product is represented by the identity.

The orthogonal complement $W^\perp$ of a subspace $W \subset V$ can now be taken relative to **any** positive definite inner (or Hermitian) product and the conclusions from Exercise 4.3 hold, namely that $V = W \oplus W^\perp$.

Let G be a **finite** group.

Definition 4.7. If V is a G-module over $\mathbb{R}$ (or $\mathbb{C}$), then the G-*invariant* inner product (or Hermitian product) on V is:

$$b(\vec{v}, \vec{w}) = \frac{1}{|G|}\sum_{g \in G} g\vec{v} \cdot g\vec{w} \qquad \left(\text{or} \quad h(\vec{v}, \vec{w}) = \frac{1}{|G|}\sum_{g \in G} \langle g\vec{v}, g\vec{w}\rangle\right)$$

Notice that these are both *positive definite*!

Proposition 4.3. The orthogonal complement $W^\perp \subset V$ of a G-submodule $W \subset V$ with respect to the G-invariant inner (or Hermitian) product on V **is** another G-submodule of V.

Proof. The G-invariance of the inner product gives:

$$b(\vec{v}, \vec{w}) = \frac{1}{|G|}\sum_{g \in G} g\vec{v} \cdot g\vec{w} = \frac{1}{|G|}\sum_{g \in G} (gh)\vec{v} \cdot (gh)\vec{w} = b(h\vec{v}, h\vec{w})$$

for any $h \in G$. Suppose $\vec{v} \in W^\perp$, i.e. $b(\vec{v}, \vec{w}) = 0$ for all $w \in W$. Then $b(h\vec{v}, \vec{w}) = b(\vec{v}, h^{-1}\vec{w}) = 0$ for all $\vec{w} \in W$. The same argument holds in the Hermitian case. ☐

Corollary 4.1. Suppose V is a vector space over $\mathbb{C}$ and $A \in \mathrm{GL}(V)$ has finite order, i.e. $A^n = \mathrm{id}$ for some n. Then A is semi-simple.

Proof. Let $G = C_n$ and let V be a G-module via $\rho(x) = A$. We know that the matrix A has (at least) one eigenvector $\vec{v} \in V$ with eigenvalue λ. The line L spanned by $\vec{v}$ is a G-submodule of V, whose orthogonal complement under the G-invariant Hermitian product is a G-submodule. The matrix A acts on the orthogonal complement $L^\perp$ with an eigenvector in $L^\perp$, and then A acts on the G-invariant orthogonal complement to that eigenvector in $L^\perp$, etc. ☐

Example 4.5. Consider the matrix from Example 4.2(c):

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}; \; A^2 = \mathrm{id}$$

Then $\mathbb{R}^2$ is a C_2-space, and the eigenvector $(1, 0)$ spans an invariant subspace. The C_2-invariant inner product is:

$$b(\vec{v}, \vec{w}) = \frac{1}{2}(\vec{v} \cdot \vec{w} + A\vec{v} \cdot A\vec{w})$$

which is represented by the matrix:

$$B = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

Then $b((1, 2), (1, 0)) = 0$ so $(1, 2)$ is "the other" eigenvector of A.

Corollary 4.2. When G is a finite group, any finite-dimensional G-module V over $\mathbb{R}$ or $\mathbb{C}$ is *completely reducible*, i.e.

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_n$$

for some set of **irreducible** G-submodules $U_1, ..., U_n \subset V$.

Proof. Either V is irreducible, and there is nothing to prove, or else V has a nonempty sub G-module $U \subset V$. With respect to the invariant dot (or Hermitian) product, the orthogonal complement $W = U^{\perp}$ is also a G-module, and $V = U \oplus W$. By induction on dimension, we may assume that U and W are direct sums of irreducible G-submodules, and so V is as well. □

Since every G-module is a direct sum of finitely many irreducible G-modules, it makes sense to try to *classify* the irreducible G-modules. We will do this over the complex numbers, where, for example, the irreducible G-modules for an abelian group were seen to be the one-dimensional characters.

As usual, let G be a finite group.

Definition 4.8. The *(left) regular* G-module over $\mathbb{C}$ is:

$$\mathbb{C}[G] = \langle e_g \mid g \in G \rangle \text{ with the action } \rho(h)(e_g) = e_{hg}$$

i.e. the vector space with one basis vector e_g for each $g \in G$, made into a G-module by permuting basis vectors by left multiplication.

This is similar to the natural representation of the permutation group, in which the vectors corresponded to elements of the set $[n]$ (rather than elements of the group $\mathrm{Perm}(n)$).

Example 4.6. Let $G = C_n$ with the ordering $\{\mathrm{id}, x, ..., x^{n-1}\}$. Then:

$$\rho(x)(e_{x^i}) = e_{x^{i+1}}$$

cyclically permutes the basis vectors, so the matrix for the action is:

$$A = \rho(x) = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Let $\zeta = e^{\frac{2\pi i}{n}}$. Then:

$$\{\vec{v}_{\zeta^m} = e_1 + \zeta^m e_2 + \cdots + \zeta^{m(n-1)} e_n \mid 0 \le m < n\}$$

is a basis of eigenvectors for A (and all powers of A), hence:

$$\mathbb{C}[C_n] = \mathbb{C}\vec{v}_{\zeta} \oplus \mathbb{C}\vec{v}_{\zeta^2} \oplus \cdots \oplus \mathbb{C}\vec{v}_{\zeta^n}$$

Notice that $\mathbb{C}\vec{v}_{\zeta^m} = \chi_{-m}$ as a one-dimensional character of C_n, since:

$$A\vec{v}_{\zeta^m} = e_2 + \zeta^m e_3 + \cdots + \zeta^{m(n-1)} e_1 = \zeta^{-m}\vec{v}_{\zeta^m}$$

and in particular, each summand in the direct sum is a *distinct* one-dimensional character, with the last being the trivial character.

Theorem 4.1. Let $\mathbb{C}[G] = \bigoplus_{i=1}^n U_i$ be a decomposition of the regular representation into irreducible G-submodules over $\mathbb{C}$. Then:

(a) Every irreducible G-module U is isomorphic to a summand.

(b) If $\dim(U) = d$, then U is isomorphic to d of the summands.

Corollary 4.3. There are only finitely many irreducible G-modules (up to isomorphism), and their dimensions $d_1, ..., d_m$ satisfy:

$$|G| = d_1^2 + d_2^2 + \cdots + d_m^2$$

Proof. Immediate from the Theorem, since $\sum_{i=1}^n \dim U_i = |G|$. □

Before we get to the proof of this, consider some examples:

(a) We've seen $\mathbb{C}[C_n]$ as a direct sum of its n characters.

(b) Since the only irreducible representations of an abelian group G are one-dimensional characters, it follows that there are $|G|$ of them, and that $\mathbb{C}[G] = \oplus\chi$ is the direct sum of the distinct characters.

(c) We've seen three irreducible complex representations of Perm(3):

trivial, sign and standard (two-dimensional)

and since $1 + 1 + 2^2 = 6 = |\text{Perm}(3)|$, there are no other irreducibles. In particular, since $\text{Perm}(3) = D_6$, the rotation/reflection irreducible representation must be isomorphic to the standard representation.

The key ingredient in this proof is Schur's Lemma, which requires us to (finally) think a bit more about the structure of the sets of linear maps between two vector spaces (or G-modules).

Observation. Let V and W be vector spaces over k. Then the set of linear maps (morphisms) from V to W is denoted by $\text{Hom}_k(V, W)$ and it is itself a vector space, via:

$$(f + g)(\vec{v}) = f(\vec{v}) + g(\vec{v}) \text{ and } (cf)(\vec{v}) = f(c\vec{v})$$

Remark. We've already seen one example, namely $V^\vee = \text{Hom}_k(V, k)$.

If $\dim(V) = m$ and $\dim(W) = n$, then $\dim(\text{Hom}_k(V, W)) = mn$. Indeed:

$$\text{Hom}_k(k^m, k^n) = \{m \times n \text{ matrices}\}$$

and the addition is the componentwise addition of matrices.

If V and W are G-modules, the set $\mathrm{Hom}_G(V, W)$ of G-linear maps is **also** a vector space. It is a subspace: $\mathrm{Hom}_G(V, W) \subset \mathrm{Hom}_k(V, W)$ since G-linear maps sum (and scalar multiply) to G-linear maps.

Schur's Lemma. (a) If V, W are irreducible G-modules, then:

$$\mathrm{Hom}_G(V, W) = \{\vec{0}\} \text{ unless } V \text{ and } W \text{ are isomorphic}$$

(b) If V, W are isomorphic G-modules **over** $\mathbb{C}$, then:

$$\dim(\mathrm{Hom}_G(V, W)) = 1$$

In particular, $\mathrm{End}_G(V) = \mathbb{C} \cdot \mathrm{id}_U$ for an irreducible G-module over $\mathbb{C}$.

Proof. The only G-submodules of an irreducible V are $\vec{0}$ and V. If $f \in \mathrm{Hom}_G(V, W)$ consider the G-submodule kernel and image of f. Either $\ker(f) = \{\vec{0}\}$ (f is injective), or $\ker(f) = V$ (f is zero). Either $\mathrm{im}(f) = \{\vec{0}\}$, ($f$ is zero) or $\mathrm{im}(f) = W$ (f is surjective). Thus either f is zero or else f is a bijection, i.e. an isomorphism. This gives (a).

For (b), suppose $f, g \in \mathrm{Hom}_G(V, W)$ are an arbitrary pair of nonzero G-linear maps. By (a) they are isomorphisms, so we may consider:

$$\mathrm{id}_W = g \circ g^{-1} \text{ and } h = f \circ g^{-1} \in \mathrm{Hom}_G(W, W)$$

Then for each scalar $\lambda \in \mathbb{C}$, we have:

$$h - \lambda \cdot \mathrm{id}_W \in \mathrm{Hom}_G(W, W)$$

and when λ is an eigenvalue of h (which happens for at least one λ), this G-linear map is **not** an isomorphism. For such a λ, then, $h - \lambda \cdot \mathrm{id}_W = 0$, hence $\lambda \cdot \mathrm{id}_W = h$ and $f = h \circ g = \lambda \cdot g$, so f is a **multiple** of g. This proves that $\mathrm{Hom}_G(V, W)$ is one-dimensional. □

Remark. Part (b) of Schur's Lemma doesn't hold if the field is $\mathbb{R}$. For example, if $A \in \mathrm{GL}(2, \mathbb{R})$ is the rotation by $2\pi/n$, then $A \in \mathrm{End}_{C_n}(\mathbb{R}^2)$ for the irreducible rotation action. Evidently $A \neq \lambda \cdot \mathrm{id}$ for any λ.

Proof of Theorem 4.1. Let U be an irreducible G-module, and let $\vec{u} \in U$ be a nonzero vector. Then the linear map:

$$f_{\vec{u}} : \mathbb{C}[G] \to U \text{ defined by } f_{\vec{u}}(e_g) = g \cdot \vec{u}$$

is G-linear, because $f_{\vec{u}}(he_g) = f_{\vec{u}}(e_{hg}) = (hg)\vec{u} = hf_{\vec{u}}(e_g)$ for all $h \in G$.
This map is not the zero map, since $f_{\vec{u}}(e_{\mathrm{id}}) = \vec{u}$. But if:

$$\mathbb{C}[G] = U_1 \oplus U_2 \oplus \cdots \oplus U_n$$

is the decomposition of $\mathbb{C}[G]$, then on each summand, $f_{\vec{u}} : U_i \to U$ is either zero or an isomorphism, by Schur's Lemma (a), so (at least) one of them is a isomorphism! This gives (a) of the Theorem.

For (b) of the Theorem, consider the linear "evaluation" map:

$$\phi : \mathrm{Hom}_G(\mathbb{C}[G], U) \to U;\ \phi(f) = f(e_{\mathrm{id}})$$

If $\phi(f) = \vec{u}$, then $f(e_g) = gf(e_{\mathrm{id}}) = g\vec{u}$, so $f = f_{\vec{u}}$ defined above! And if $\phi(f) = \vec{0}$, then $f(e_g) = \vec{0}$ for all $g \in G$, and f is the zero map. So ϕ is an isomorphism.

This means in particular that $\dim(\mathrm{Hom}_G(\mathbb{C}[G], U)) = \dim(U)$. But by Schur Lemma (b), the dimension of $\mathrm{Hom}_G(U_i, U) = 1$ for each irreducible summand of $\mathbb{C}[G]$ that is isomorphic to U, and it follows that there are $\dim(U)$ of them. $\square$

Exercises.

4.1. Prove that $V = U \oplus W$ for subspaces $U, W \subset V$ if and only if:

(i) $U \cap W = \vec{0}$ and (ii) $\dim(U) + \dim(W) = \dim(V)$.

4.2. (a) Show that G-modules with G-linear maps are a category.

(b) Show that the kernel and image of a G-linear map are G-modules.

(c) Show that the space of cosets of a G-submodule is a G-module.

4.3. Prove that the orthogonal complement of a subspace $U \subset V$ of a vector spaces over $\mathbb{R}$ or $\mathbb{C}$ satisfies: $V = U \oplus U^\perp$.

One cheap way to get new irreducible representations from old ones is to "twist by a character," i.e. given representations $\rho : G \to \mathrm{GL}(U)$ and $\chi : G \to \mathbb{C}^*$, consider $\rho \otimes \chi \in \mathrm{GL}(U)$ defined by $g\vec{u} = \chi(\vec{u}) \cdot \rho(\vec{u})$.

4.5. What conditions on ρ and χ guarantee that the G-module on U for $\rho \otimes \chi$ is **not** isomorphic to the G-module for ρ?

4.6. Find all the irreducible representations of:

(a) D_8 and D_{10} (do you see a pattern?)

(b) Alt(4) (use the map $\mathrm{Alt}(4) \to C_3 = \mathrm{Alt}(4)/K_4$.)

(c) Perm(4) (use the map $\mathrm{Perm}(4) \to \mathrm{Perm}(3) = \mathrm{Perm}(4)/K_4$)

(d) Alt(5)

The *trace* $tr(\rho(g))$ of an irreducible representation ρ evaluated at an element $g \in G$ is the same for all elements of the same conjugacy class. This is not surprising. It is a consequence of Exercise 2.9. What **is** surprising is that the number of irreducible representations is equal to the number of conjugacy classes, and:

• There is a Hermitian inner product on the complex vector space $V = \oplus_c \mathbb{C} \cdot e_c$ of dimension equal to the number of conjugacy classes c in the group G (which is the number of irreducible representations) such that the "trace" vectors of irreducible representations:

$$\vec{v}_\rho = \sum_c \mathrm{tr}(\rho(g_c))e_c \text{ for arbitrary choices of } g_c \in c$$

are an orthonormal basis of V (length one and mutually orthogonal).

The matrix with column vectors v_ρ is the *character table* of G.

4.7. Figure out the Hermitian inner product that makes this happen. (Hint: It involves the numbers elements in each conjugacy class) and compute the character tables of cyclic groups and the groups whose representations you found in 4.6.