Lecture 7. Functions and Stuff

PCMI Summer 2015 Undergraduate Lectures on Flag Varieties

Lecture 7. Functions and differentiable and analytic manifolds. The implicit function theorem, cotangent spaces and Lie algebras.

Consider first the hierarchy for functions $f : \mathbb{R} \to \mathbb{R}$ on the real line.

Continuous functions (at x_0) are those with the property that:

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that if } |x - x_0| < \delta \text{ then } |f(x) - y_0| < \epsilon$$

i.e. $(x_0 - \delta, x_0 + \delta) \subset f^{-1}(y_0 - \epsilon, y_0 + \epsilon)$ where $y_0 = f(x_0)$.

Or, more succinctly,

$$\lim_{x \to x_0} f(x) = y_0 \text{ and } y_0 = f(x_0)$$

Of course, this matches the topological definition from Lecture 6.

Differentiable functions (at x_0) are continuous functions that are well-approximated by linear functions, in the sense that:

$$\lim_{h \to 0} \frac{|f(x_0 + h) - (y_1 h + y_0)|}{|h|} = 0$$

for (uniquely determined) real numbers $y_0 = f(x_0)$ and $y_1 := f'(x_0)$. The function f is *continuously differentiable* on an interval (a, b) if the derivative $f'(x)$ is continuous on (a, b). We will ignore the infinite hierarchy of functions that have n but not $n+1$ continuous derivatives, and pass directly to *infinitely differentiable* functions, whose derivatives determine a **Taylor series** approximation to f near x_0:

$$f(x) \approx y_0 + y_1(x - x_0) + \frac{y_2}{2!}(x - x_0)^2 + \cdots$$

where $y_n = f^{(n)}(x_0)$ is the nth derivative evaluated at x_0.

Analytic functions on (a, b) are infinitely differentiable *and agree with their Taylor series*, in the sense that their Taylor series converge to $f(x_0)$ at each point $x_0 \in (a, b)$.

Rational functions are ratios of polynomials. Not only are they analytic (on their domain), but the Taylor series can be obtained by algebraically inverting the denominator polynomial as a power series.

Functions of each type may be added, subtracted, multiplied and divided (where nonzero), remaining in the same class, and *compositions* of functions of a given type remain of the same type.

The classes are distinct. There is a *huge* gap between each class of functions. Some standard functions that distinguish the classes include:

(a) $f(x) = |x|$ (continuous, not differentiable at 0)

(b) $f(x) = e^{-1/x^2}, f(0) = 0$ (the Taylor series at 0 is identically 0).

(c) $e^x, \ln(x), \sin(x), \cos(x)$ are analytic but not rational.

When the same hierarchy is imposed on functions $f : \mathbb{C} \to \mathbb{C}$ of one **complex** variable, however, the distinction between differentiable and analytic vanishes in a striking way. Continuity is defined as for functions of two real variables (see below), but differentiability in a complex variable has extraordinary consequences.

Definition 7.1. $f(z) = u(z) + iv(z) : \mathbb{C} \to \mathbb{C}$ is *holomorphic at* z_0 if:

$$\lim_{z \to z_0} \frac{|f(z) - (w_1 z + w_0)|}{|z - z_0|} = 0$$

for complex numbers $w_0 = f(z_0)$ and $w_1 := f'(z_0)$.

Such functions satisfy the *Cauchy-Riemann* equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which imply that u and v are *harmonic* functions in x and y, satisfying:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

but also that $|f| = \sqrt{u^2 + v^2}$ is a harmonic function of x and y. This has the following important consequence:

Maximum Principle. If $f(z)$ is a holomorphic function, then $|f(z)|$ does not have a local maximum at a point of a ball in its domain unless $f(z)$ is constant on the ball.

The Cauchy integral formula yields another important consequence:

(Cauchy) Theorem. If f has one complex derivative near z_0, then f is infinitely differentiable **and** (complex) analytic near z_0.

So there is no "gap" between holomorphic (complex differentiable) functions and analytic functions of a complex variable. It is surprising that having one complex derivative is so much more consequential than having a real derivative!

This discussion is easily promoted to functions of several variables.

Continuity. $f : \mathbb{R}^n \to \mathbb{R}$ is *continuous* at $\vec{x}_0$ if:

$$\lim_{\vec{x} \to \vec{x}_0} |f(\vec{x}) - y_0| = 0 \text{ for } y_0 = f(\vec{x}_0)$$

(agreeing with the definition via the Euclidean topology)

Differentiability. A differentiable function is well-approximated by a linear function:

$$\lim_{\vec{x}\to\vec{x}_0} \frac{|f(\vec{x}) - (\vec{y_1}\cdot\vec{x} + y_0)|}{|\vec{x} - \vec{x}_0|} = 0$$

for $y_0 = f(\vec{x}_0)$ and

$$\vec{y}_1 := \left(\frac{\partial f}{\partial x_1}(\vec{x}_0), \ldots, \frac{\partial f}{\partial x_n}(\vec{x}_0)\right) \text{ the } \textit{gradient}$$

where the partial derivatives of f are the derivatives in each x_i direction, holding the others variables constant.

Remark. The existence (and continuity) of partial derivatives are not in general enough to imply differentiability.

An infinitely differentiable function f has a Taylor expansion:

$$f(x_1, ..., x_n) \approx \sum_{D=(d_1,...,d_n)} \frac{1}{d_1!\cdots d_n!}\frac{\partial^D f}{(\partial x)^D}(\vec{x}_0)(x - x_0)^D$$

near $\vec{x}_0$ and f is **analytic** if it agrees with its Taylor approximation.

Rational. A rational function is a ratio of polynomials.

Once again, there is a pleasant surprise when we replace $\mathbb{R}$ with $\mathbb{C}$ (and say "holomorphic" instead of "differentiable").

Hartog's Theorem. A function $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic if and only if f is holomorphic in each variable separately.

Finally, the type of a **map** $f : \mathbb{R}^n \to \mathbb{R}^m$ is determined by the coordinate functions $f_i = y_i \circ f$ for $y_i : \mathbb{R}^m \to \mathbb{R}$, and a composition and inverse of (invertible) maps preserves the type. This is a consequence of the chain rule for differentiable maps, and algebraic manipulations for analytic and rational maps.

Next, we turn to manifolds. A manifold M is **differentiable** if there is a consistent notion of (infinitely) differentiable functions among the continuous functions $f : U \to \mathbb{R}$ on open subsets of M. This is captured in the structure of a *sheaf of infinitely differentiable functions* on M, but before we jump through that hoop, consider the important example:

Gluing. Suppose $U_1, ..., U_m \subset \mathbb{R}^n$ are open subsets, equipped with gluing data $U_{ij} \subset U_i$ and $f_{ij} : U_{ij} \to U_{ji}$ that satisfies the gluing criteria. If the $f_{ij} : U_{ij} \to U_{ji} \subset \mathbb{R}^n$ are each infinitely differentiable maps, thought of as maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, then the glued manifold M is a differentiable manifold. If $U_1, ..., U_m \subset \mathbb{C}^n$ and the gluing is by holomorphic maps, then the manifold is holomorphic. In other words, if the glue has a property, the manifold has that property.

Example 7.1. (a) Consider again the sphere from Lecture 6, obtained by gluing $\mathbb{R}^n - 0$ to $\mathbb{R}^n - 0$, with (self-inverse) gluing map:

$$f_{12}(x_0, ..., x_n) = \left(\frac{x_0}{x_0^2 + \cdots + x_n^2}, \ldots, \frac{x_n}{x_0^2 + \cdots + x_n^2} \right)$$

These gluing functions are differentiable, analytic and even rational. If $y \in U_2$, then $g(y_0, ..., y_n)$ is infinitely differentiable from the U_2 point of view if it is infinitely differentiable in y variables. From the U_1 point of view, g becomes $g(x_0/|x|^2, ..., x_n/|x|^2) = g \circ f_{12}$ as a function of the x-variables, and we then ask whether this is infinitely differentiable in the x-variables. Evidently, differentiability of the glue is necessary for the points of view to coincide in general.

Notice, however, that if we replace $\mathbb{R}^n$ with $\mathbb{C}^n$, then the glue:

$$f_{12}(z_0, ..., z_n) = \left(\frac{z_0}{z\overline{z}}, \ldots, \frac{z_n}{z\overline{z}} \right)$$

is **not** holomorphic, and indeed when $n > 1$, there is no way to create a compact holomorphic manifold by adding a point to $\mathbb{C}^n$.

(b) Consider projective space $\mathbb{P}^n$ from Lecture 6. This was covered by open sets U_i with gluing data:

$$f_{ij}(x_{0|i}, ..., x_{n|i}) = (x_{0|i}/x_{j|i}, ..., x_{n|i}/x_{n|j})$$

These are rational functions, and they remain rational when $\mathbb{R}$ is replaced by $\mathbb{C}$. Thus, not only is $\mathbb{P}^n_{\mathbb{R}}$ a differentiable manifold, but $\mathbb{P}^n_{\mathbb{C}}$ is a compact holomorphic manifold, and even an *algebraic* manifold. Moreover, this holds for the Grassmannian as well.

Remark. There are many, many differentiable functions on the sphere. Since the gluing for the sphere compactification of $\mathbb{C}^n$ is not holomorphic, it doesn't even make sense to ask for holomorphic functions on this sphere. It **does** make sense, however, to ask for holomorphic functions on $\mathbb{P}^n_{\mathbb{C}}$ or $\mathbb{G}(m, n)$ over $\mathbb{C}$. The answer is somewhat sobering.

Proposition 7.1. The only holomorphic functions $f : M \to \mathbb{C}$ on a compact, holomorphic connected manifold are the constants.

Proof. The image $f(M) \subset \mathbb{C}$ of a holomorphic function is compact. By the Heine-Borel theorem, $f(M)$ is therefore closed and bounded, so in particular, $|f(z)|$ attains a maximum value for $z \in M$. Suppose $f(z_0) = c \in \mathbb{C}$ has this maximum length. Then the set $f^{-1}(c) \subseteq M$ is closed by continuity, and open by the maximum principle! $\square$

Remark. The original maximum principle was for functions of one complex variable. This is for several. It follows. Try to figure out why.

Implicit Function Theorem. Let $f_1, ..., f_m$ be analytic functions in n variables (real or complex) for some $m \leq n$. This gives a system:

$$f_1(x_1, ..., x_n) = \cdots = f_m(x_1, ..., x_n) = 0$$

of equations and a **set** $M = V(f_1, ..., f_n) \subset \mathbb{R}^n$ (or $\mathbb{C}^n$) of solutions of the system of equations. Suppose that for each $p \in M$, the *Jacobian matrix of partial derivatives:*

$$Jac(p) = \left(\frac{\partial f_i}{\partial x_j}\right)(p)$$

has an invertible $m \times m$ minor. Then M is an analytic manifold.

Precisely, suppose $p = (p_1, ..., p_n) \in M$ and $J \subset [n]$ indexes a minor of the Jacobian that is invertible at the point p. Then in a small enough neighborhood $B_r(p) \cap M$, the set M coincides with the **graph** of an analytic mapping:

$$\Phi : \mathbb{R}^{n-m} \to \mathbb{R}^m \textit{ in the complementary coordinates}$$

i.e. the coordinates $x_{j_1}, ..., x_{j_m}$ on M are analytic functions of the complementary set of (free) coordinates in a neighborhood of $x \in M$.

Example 7.2. Consider the circle $S^1 = \{x^2 + y^2 - 1 = 0\} \subset \mathbb{R}^2$.

The gradient of the function $f(x, y) = x^2 + y^2 - 1$ is $\nabla f = (2x, 2y)$ which vanishes at no point of S^1. Near the point $(0, 1)$, the second partial doesn't vanish, and the circle is locally the graph of the function:

$$\Phi_1 : \mathbb{R} \to \mathbb{R};\ \Phi_1(x) = \sqrt{1 - x^2}$$

in the complementary variable. Near the point $(0, -1)$, again the second partial doesn't vanish, and the circle is locally the graph of the function:

$$\Phi_2 : \mathbb{R} \to \mathbb{R};\ \Phi_2(x) = -\sqrt{1 - x^2}$$

Near the point $(1, 0)$, the first partial doesn't vanish, and the circle locally the graph of the function:

$$\Phi_3 : \mathbb{R} \to \mathbb{R};\ \Phi_3(y) = \sqrt{1 - y^2}$$

and finally, near the point $(-1, 0)$, the first partial fails to vanish, and the circle is locally the graph of the functions:

$$\Phi_4 : \mathbb{R} \to \mathbb{R};\ \Phi_4(y) = -\sqrt{1 - y^2}$$

All together, the domains of these four functions cover the circle with local graphs of analytic functions. The circle can now be **glued** from these four domains, with analytic glue. Notice that although the equation was **polynomial**, the manifold is "only" analytic.

Corollary 7.1. Suppose $f_1, ..., f_m$ are analytic functions in n variables, and $M \subset V(f_1, ..., f_m)$ consists of of all the points at which the Jacobian matrix has rank m. Then U is an analytic manifold of dimension $n-m$. If the variables were **complex**, then M has **complex** dimension $n-m$, i.e. real dimension $2(n-m)$.

Proof. By the implicit function theorem, M can assembled by gluing balls in $\mathbb{R}^{n-m}$ (or $\mathbb{C}^{n-m}$) with analytic glue.

Let's go back to our Lie groups and apply this neat trick:

$$\mathrm{SL}(n, \mathbb{C}) = V(\det(x_{ij}) - 1) \subset \mathbb{C}^{n^2}$$

What's the gradient of the determinant? By expanding the determinant along a row or column, each partial derivative of the determinant is "clearly" the complementary minor:

$$\frac{\partial \det}{\partial x_{ij}} = (-1)^{i+j} \det(M_{ij})$$

But if **all** of these vanish, then the determinant is zero, so they **don't** all vanish at any point of the special linear group!

Conclusion. $\mathrm{SL}(n, \mathbb{C})$ is an analytic manifold of dimension $2(n^2-1)$. (Dimension n^2-1 as a complex manifold).

Next up, the orthogonal group:

$$O(n, \mathbb{C}) = V(A^T A - \mathrm{id}) \subset \mathbb{C}^{n^2}$$

This is a system of $\binom{n+1}{2}$ quadratic polynomial equations, registering the fact that each column dots with itself to one (n equations) and that different columns dot with each other to zero ($\binom{n}{2}$ equations). What's the Jacobian matrix of all of these $\binom{n+1}{2}$ quadratic polynomials? Why does it have rank equal to the number of equations at $A \in O(n, \mathbb{C})$? There has to be a better way than just cranking out the derivatives.

There is a better way. Let's reconsider what partial derivatives are supposed to accomplish. In fact, let's consider a system of equations:

$$f_1(x_1, ..., x_n) = ... = f_m(x_1, ..., x_n) = 0$$

as one single (vector valued) equation $\vec{f}(\vec{x}) = \vec{0}$. The differentiability of all the functions goes into one single "master" equation as:

$$\vec{f}(p + \epsilon\vec{x}) = \vec{f}(p) + \epsilon Jac(p)\vec{x} + O(\epsilon^2)$$

The Jacobian $Jac(p)$ then has rank m if the map:

$$Jac(p) : \mathbb{C}^n \to \mathbb{C}^m \text{ is surjective}$$

or, equivalently, if the kernel has dimension $n-m$.

Applying this to the equations of $O(n,\mathbb{C})$ at $A \in O(n,\mathbb{C})$ gives:

$$(A+\epsilon B)^T(A+\epsilon B) - \text{id} = \epsilon(A^TB + B^TA) + O(\epsilon^2)$$

from which we conclude that the Jacobian $Jac(A)$ is the operator:

$$Jac(A)(B) = A^TB + B^TA$$

When $A = \text{id}$, this operator is:

$$Jac(\text{id})(B) = B + B^T$$

which has as its image equal to the space of symmetric matrices, and the symmetric matrices are a vector space of dimension equal to $\binom{n+1}{2}$, which was the number of equations! So $O(n,\mathbb{C})$ is an analytic manifold in some neighborhood of id. In fact, it is an analytic manifold everywhere. Instead of proving this by analyzing $Jac(A)$, you should think about how to prove it using the fact that $O(n,\mathbb{C})$ is a group.

This is cool! Let's apply it back to the special linear group:

$$\det(\text{id} + \epsilon B) - 1 = \epsilon Tr(B) + O(\epsilon^2)$$

so $Jac(\text{id})(B) = Tr(B)$ is the trace operator in this case. The image of the trace is one dimensional, so this is another proof that $\text{SL}(n,\mathbb{C})$ is an analytic manifold, at least in a neighborhood of the identity.

You've been promised a definition for a differentiable manifold, but you've only gotten examples. It's time to follow through. We will think categorically, and start with:

Definition 7.2. The *category of open subsets* of a manifold M has:

- Open subsets $U \subset M$ as the objects.
- Inclusions $U \subset V$ (of open sets) as the morphisms.

Remark. Compare this to the category of subsets of a (finite) set.

To each open set $U \subset M$, we associate the **ring**:

$$\mathcal{C}_M(U) = \{\text{continuous functions } f : U \to \mathbb{R}\}$$

Definition 7.3. A commutative ring with 1 (or simply ring) satisfies all the properties of a field except for multiplicative inverses. A *ring homomorphism* is a map $\phi : R \to S$ of rings such that $\phi(1) = 1$, $\phi(r_1 + r_2) = \phi(r_1) + \phi(r_2)$ and $\phi(r_1r_2) = \phi(r_1)\phi(r_2)$ for all $r_1, r_2 \in R$.

Moment of Zen. The category $\mathcal{C}Rings$ consists of:

(a) Commutative rings with 1, (b) Ring homomorphisms

We will say a lot more about commutative rings in Lecture 9.

These rings of continuous functions satisfy the following:

(1) They restrict. Given a morphism $U \subset V$ in the category of open sets, "restriction of functions from V to U" is a ring homomorphism:

$$\rho_{V,U} : \mathcal{C}_M(V) \to \mathcal{C}_M(U)$$

i.e. a morphism in the category of rings.

(2) Locally zero means globally zero. If $f \in \mathcal{C}_M(U)$ and $U = \cup U_\lambda$ is an open cover of U with the property that:

$$\rho_{U,U_\lambda}(f) = 0 \in \mathcal{C}_M(U_\lambda) \text{ for all } \lambda$$

then $f = 0$ is the zero function (on U).

(3) They patch. If $U = \cup U_\lambda$, and $f_\lambda \in \mathcal{C}(U_\lambda)$ satisfy:

$$\rho_{U_\lambda, U_\lambda \cap U_\mu}(f_\lambda) = \rho_{U_\mu, U_\lambda \cap U_\mu}(f_\mu)$$

for all pairs λ, μ, then the f_λ patch to a global function:

$$f \in \mathcal{C}_M(U) \text{ such that } \rho_{U,U_\lambda}(f) = f_\lambda \text{ for all } \lambda$$

Remark. Condition (1) **precisely** says that:

$$\mathcal{C}_M : \text{category of open sets in } M \to \mathcal{C}Rings$$

is a **functor** from the category of open sets in M to the category of commutative rings! Conditions (2) and (3) are the (famous) additional conditions that an abstract functor (pre-sheaf) from open sets to rings (or abelian groups) needs to satisfy in order to be a **sheaf**.

Roughly, a differentiable manifold is a manifold for which the sheaf of continuous functions has a sensible subsheaf of differentiable functions. As a warmup, consider:

Example 7.3. The sheaf $\mathbb{Z}_M$ of *locally constant* integer-valued functions on a manifold M is the sub sheaf of $\mathcal{C}$ defined by:

$$\mathbb{Z}_M(U) = \{\text{continuous functions } f : U \to \mathbb{Z}\}$$

with the same restriction and gluing properties as $\mathcal{C}_M$. Notice that this is a very discrete object! If U is connected, then $\mathbb{Z}_M(U) = \mathbb{Z}$, because the only continuous functions from a connected set to the integers are the constant functions! We can similarly define sheaves $\mathbb{R}_M$ and $\mathbb{C}_M$ to be the subsheaves of functions that are locally constant as maps $f : U \to \mathbb{R}$ (or $\mathbb{C}$) (i.e. continuous for the discrete topologies!).

Remark. These simple-minded locally constant sheaves are actually very interesting! Within them is encoded all the information of the *singular cohomology ring* of the manifold M, betti numbers, etc.

Here is a very important observation.

Suppose $\phi : M \to N$ is a continuous map of manifolds. Then there is a "pull-back" map on continuous functions:

$$\phi^* : \mathcal{C}_N(U) \to \mathcal{C}_M(f^{-1}(U)); \quad \phi^*(f) = \phi \circ f$$

that commutes with the restrictions. Notice also that the pull-back **respects** the subsheaves of locally constant functions in the sense that:

$$\phi^* : \mathbb{Z}(U) \to \mathbb{Z}(f^{-1}(U))$$

In the special case of the manifold $\mathbb{R}^n$ itself, or any open subset $M \subset \mathbb{R}^n$, there is another subsheaf, namely the sheaf of *infinitely differentiable* functions $\mathcal{C}_M^\infty$, defined by:

$$\mathcal{C}_M^\infty(U) = \{\text{infinitely differentiable functions } f : U \to \mathbb{R}\}$$

that sits **between** the locally constant and the continuous functions.

Definition 7.4. A manifold M *together with a subsheaf* $\mathcal{C}_M^\infty \subset \mathcal{C}_M$ of the sheaf of continuous functions is **differentiable** if there is a "chart" consisting of an open cover $M = \cup U_\lambda$ together with homeomorphisms:

$$f_\lambda : B_\lambda \to U_\lambda; \text{ with balls } B_\lambda \subset \mathbb{R}^n$$

such that $f^*(\mathcal{C}_M^\infty(V)) = \mathcal{C}_{B_\lambda}^\infty(f^{-1}(V))$ for all open subsets $V \subset U_\lambda$.

In other words, there should exist a cover of M by open balls and a notion of differentiability of functions (captured in the sheaf) such that a function is (globally) differentiable if and only if it is (locally) differentiable on each of the open balls.

Moment of Zen. the category $\mathcal{D}$iff consists of:

(a) Differentiable manfolds, (b) Differentiable maps.

Sheaves allow us to effortless define the morphisms:

Definition 7.5. A continuous map $\phi : M \to N$ of differentiable manifolds $(M, \mathcal{C}_M^\infty)$ and $(N, \mathcal{C}_N^\infty)$ is **differentiable** if:

$$\phi^*(\mathcal{C}_N^\infty(U)) \subset \mathcal{C}_M^\infty(\phi^{-1}(U)) \text{ for all } U \subset N$$

i.e. if the pull-back respects differentiable functions!

Vocabulary. An isomorphism in $\mathcal{D}$iff is called a diffeomorphism.

Definition 7.6. A **Lie Group** G is a group within the category of differentiable manifolds, i.e. G is a differentiable manifold, and:

$$m : G \times G \to G \text{ and } i : G \to G$$

are differentiable maps.

Note. Analytic and holomorphic manifolds are defined the **same way**!

Still (you might wonder), what does all this buy us?

Differentiable Manifolds have Tangent Bundles! More precisely, there is an intrinsic notion of a tangent space T_pM to a differentiable manfold M at a point $p \in M$ with the following properties:

(a) The tangent space to $\mathbb{R}^n$ at any point is $\mathbb{R}^n$ (no variation).

(a) In general, tangent spaces "vary differentiably" as p varies in M.

(b) Tangent spaces push forward. A differentiable map $\phi : M \to N$ induces linear maps $\phi_* : T_pM \to T_{\phi(p)}N$ for each point $p \in M$.

(c) If $M \subset \mathbb{R}^n$ is the locus of zeroes of a system of m equations with Jacobian matrix of rank m at p, then the **extrinsic tangent space**, defined as the kernel of the Jacobian matrix $Jac(p) \subset \mathbb{R}^n = T_p\mathbb{R}^n$, is the image of the intrinsic tangent space $i_*T_pM \subset \mathbb{R}^n$ under the push forward by the inclusion map $i : M \subset \mathbb{R}^n$.

(d) If G is a Lie group, then the action of G on itself by conjugation $c : G \to \mathrm{Aut}(G)$ produces diffeomorphisms $c(g) : G \to G$, each of which fixes the identity, and therefore induces an "adjoint" action of the **tangent space at the identity** on itself via $c(g)_* : T_{\mathrm{id}}G \to T_{\mathrm{id}}G$ This gives the tangent space to a Lie group at the identity the structure of a *Lie algebra*.

Let's recap where we are with our Lie groups:

$\mathrm{GL}(n, \mathbb{C})$ (or $\mathbb{R}$). Not compact, defined as the complement of a zero set of the determinant. Tangent space at id is **all** $n \times n$ matrices.

$\mathrm{SL}(n, \mathbb{C})$ (or $\mathbb{R}$). Not compact, defined as the zero set of $\det(A) - 1$. Tangent space at id is all **trace zero** matrices.

$O(n, \mathbb{C})$. Not compact, defined as the zero set of $A^TA - \mathrm{id}$. Tangent space at id is all skew symmetric matrices (in particular, traceless!).

$O(n, \mathbb{R})$. Compact, zero set of $A^TA - \mathrm{id}$. Tangent space at id is all real skew-symmetric matrices.

$U(n)$. Compact, zero set of $A^T\overline{A} - \mathrm{id}$. Tangent space at id is all skew-Hermitian matrices ($B^T = -\overline{B}$). This forces the diagonal to be purely imaginary. The additional condition for $SU(n)$ is the traceless condition.

Exercises.

1. State the maximum principle for holomorphic functions of several variables and deduce it from the one-variable maximum principle.

2. A mini-étude on the exponential map:

(a) Compute the tangent space of $U(n)$ and deduce what skew-Hermitian means.

(b) Compute the tangent space of $U(1)$ and notice that by exponentiating a tangent vector, we can get $T_{\mathrm{id}}U(1)$ to map onto $U(1)$.

(c) Do the same for $\mathrm{SO}(2,\mathbb{R})$. Here the tangent vector is a two-by-two matrix, and we are exponentiating it as a matrix.

(d) Do the same for $\mathrm{SU}(2)$. This should blow your mind.

(e) Is the exponential of an **arbitrary** matrix well defined? If so, can you show that the exponential of a matrix taken from the tangent space to any of our Lie groups lands in the Lie group?

3. The **Lie bracket**, denoted $[*,*]$, is a bilinear operation:

$$[*,*] : V \times V \to V$$

on a vector space V that anti-symmetric:

$$[X,Y] = -[Y,X]$$

and satisfies the *Jacobi identity:*

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = \vec{0}$$

(a) Show that the operation:

$$[X,Y] = XY - YX$$

is a Lie bracket on matrices. Give an example to show that this is **not**, in general, associative!

(b) Show that each of our tangent spaces to Lie groups is closed under the Lie bracket on $n \times n$ matrices.

4. Let's go after the meaning of a tangent vector. We can put an equivalence relation on differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ defined in a neighborhood of the origin (or "germs of functions at the origin") by:

$$f \sim g \leftrightarrow \nabla f(\vec{0}) = \nabla g(\vec{0})$$

i.e. if their gradients at the origin are the same. The gradient is not well-defined if $\vec{0} \in \mathbb{R}^n$ is replaced by $p \in M$, a point of a differentiable manifold, but the **equivalence classes** remain the same, and we get a well-defined map:

$$d : \mathcal{C}^\infty_M(U) \to T^*_pM;\ df = [f - f(p)]$$

from differentiable functions to the equivalence class of differentiable functions under this equivalence relation. These equivalence classes add! In fact, they are "just" the dual space to $\mathbb{R}^n$, generated by dx_i, whenever $x_1, ..., x_n$ is a "system of coordinates" near p, i.e. a collection of differentiable functions on a neighborhood of p that maps the neighborhood diffeormorphically to a ball in $\mathbb{R}^n$. A cotangent vector is just one of these equivalence classes. A tangent vector is an element of the dual vector space.

If $\phi : (-\epsilon, \epsilon) \to M$ is a "short differentiable curve," with $\phi(0) = 0$, then the curve determines a tangent vector $\phi'(0)$ (i.e. a dual vector to the cotangent space) at p, via:

$$\phi'(0)(df) = \frac{d(f \circ \phi)}{dt}|_{t=0}$$

In other words, in the dual coordinates $\frac{\partial}{\partial x_i}$ to the dx_i, we have:

$$\phi'(t) = \sum_{i=1}^{n} \frac{d\phi_i(t)}{dt}\frac{\partial}{\partial x_i}$$

which is the usual form for the vector field associated to ϕ.

Apply this to the exponential map:

$$\phi_A(t) = e^{At}$$

to explicitly see the extrinsic tangent vector to the Lie groups as tangent vectors. In general, try to understand why tangent spaces push forward, using this description of the tangent space.