Lecture 9. Complex Affine Varieties

PCMI Summer 2015 Undergraduate Lectures on Flag Varieties

Lecture 9. When we think of the Lie groups $\mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C}), \mathrm{SO}(n,\mathbb{C}), \mathrm{Sp}(2n,\mathbb{C})$ over the complex numbers as algebraic groups, we may forget the groups entirely and replace them with their $\mathbb{C}$-algebras of regular functions. Within these regular functions the irreducible representations of the Lie groups will eventually be found.

Motivating Example. Because the circle $U(1)$ is abelian, all of its irreducible representations are one-dimensional characters, and it is easy to see that the continuous characters of $U(1)$ are the powers:

$$\chi_n(e^{i\theta}) \cdot v = e^{in\theta} \cdot v$$

On the other hand, consider $\mathrm{GL}(1,\mathbb{C}) = \mathbb{C}^*$. The regular (algebraic) functions on $\mathbb{C}^*$ are Laurent polynomials in the variable z, which are organized by degree:

$$\mathbb{C}[z, z^{-1}] = \bigoplus_{n=-\infty}^{\infty} \mathbb{C} \cdot z^n$$

But when we act on these by $\mathbb{C}^*$ (with the variable w) by pulling back under the left multiplication, we get:

$$w \cdot z^n = (w^{-1}z)^n = w^{-n}z^n$$

which is just the one-dimensional representation χ_{-n} extended to $\mathbb{C}^*$.

In other words, the algebraic functions on $\mathbb{C}^*$ break up precisely into a complete set of characters of $\mathbb{C}^*$, which agree with the continuous characters of the circle $U(1)$. Our goal is to see how this idea extends to the non-abelian Lie groups, using some algebraic geometry.

Definition 9.1. (a) A $\mathbb{C}$-algebra A is a commutative ring with $1 \in A$ together with an inclusion of $\mathbb{C} \subset A$ as a field of "scalars."

(b) A ring homomorphism $f : A \to B$ of $\mathbb{C}$-algebras that is also the identity map on scalars is a $\mathbb{C}$-algebra homomorphism. When the context is clear, we will just say that f is a homomorphism.

Example 9.1 The polynomial rings $A = \mathbb{C}[x_1, ..., x_n]$ are $\mathbb{C}$-algebras.

Definition 9.2: A (proper) *ideal* is a subset $0 \in I \subset A$ that is:

(a) Closed under (internal) addition: $b_1, b_2 \in I \Rightarrow b_1 + b_2 \in I$.

(b) Closed under (external) multiplication: $a \in A, b \in I \Rightarrow ab \in I$.

(c) Satisfies $I \cap \mathbb{C} = 0$ (otherwise $I = A$ is improper).

Remark. An ideal in A is **not** a subring of I, because $1 \notin I$.

Proposition 9.1. The kernel of a homomorphism $f : A \to B$:

$$\ker(f) = \{a \in A \mid f(a) = 0\}$$

is an ideal and conversely, if $I \subset A$ is a (proper) ideal, then A/I is a $\mathbb{C}$-algebra and $q : A \to A/I$ is a $\mathbb{C}$-homomorphism with kernel I.

Proof. Checking that the kernel is an ideal is straightforward. In the other direction, the interesting operation is coset multiplication:

$$(a_1 + I)(a_2 + I) = (a_1a_2 + a_1I + a_2I + I^2) = a_1a_2 + I$$

because of the closedness properties of an ideal. □

There is a *bijection of ideals* under $q : A \to A/I$

$$q^{-1} : \{\text{ideals in } A/I\} \to \{\text{ideals in } A \text{ that contain } I\}$$

An ideal $m \subset A$ is *maximal* if it is contained in no (proper) ideal.

Exercise 9.1. A $\mathbb{C}$-algebra with no proper ideal other than 0 is a field. The quotient of a $\mathbb{C}$-algebra by a maximal ideal is a field.

There are dueling notions of finite-generatedness for algebras and ideals:

Definition 9.3. (a) A $\mathbb{C}$-algebra A is *finitely generated* over $\mathbb{C}$ if there is a surjective $\mathbb{C}$-algebra homomorphism from a polynomial ring:

$$\phi : \mathbb{C}[x_1, ..., x_n] \to A$$

in which case the images $y_i = \phi(x_i)$ are *generators* of A (over $\mathbb{C}$).

(b) An *ideal* $I \subset A$ is (finitely) generated by $b_1, ..., b_n \in I$ if:

$$I = \langle b_1, ..., b_n \rangle = \{b_1a_1 + \cdots + b_na_n \, | a_1, ..., a_n \in A\}$$

Let's quickly relate this to algebraic groups:

Definition 9.4. The *algebraic functions* on the algebraic groups are:

(a) $\mathbb{C}[\mathrm{SL}(n, \mathbb{C})] = \mathbb{C}[x_{ij}]/\langle \det(X) - 1 \rangle$.

(b) $\mathbb{C}[\mathrm{GL}(n, \mathbb{C})] = \mathbb{C}[x_{ij}, y]/\langle y \det(X) - 1 \rangle$

(c) $\mathbb{C}[O(n, \mathbb{C})] = \mathbb{C}[x_{ij}]/\langle XX^T - \mathrm{id} \rangle$

(d) $\mathbb{C}[\mathrm{SO}(n, \mathbb{C})] = \mathbb{C}[x_{ij}]/\langle XX^T - \mathrm{id}, \det(X) - 1 \rangle$.

where $X = (x_{ij})$ and $XX^T - \mathrm{id}$ is a system of n^2 polynomials.

Example 9.2. The algebraic functions on $\mathrm{SO}(2, \mathbb{C})$ are:

$$\mathbb{C}[x, y, z, w]/\langle x^2 + y^2 - 1, xz - yw, z^2 + w^2 - 1, xw - yz - 1 \rangle$$

Two theorems of Hilbert are crucial:

Basis Theorem: Every ideal in $\mathbb{C}[x_1, ..., x_n]$ is finitely generated.

Nullstellensatz. The maximal ideals in $\mathbb{C}[x_1, ..., x_n]$ are exactly:

$$m_p = \ker(e_p) = \langle x_1 - p_1, ..., x_n - p_n \rangle \text{ for } p = (p_1, ..., p_n) \in \mathbb{C}^n$$

where $e_p : \mathbb{C}[x_1, ..., x_n] \to \mathbb{C}$ is "evaluation at the point p."

Definition 9.5. Let A be a finitely generated $\mathbb{C}$-algebra. Then:

$$\text{mspec}A = \{\text{maximal ideals } m \subset A\}$$

is the *maximum spectrum of* A. This is intrinsic to A, but if generators:

$$\phi : \mathbb{C}[x_1, ..., x_n] \to A$$

for the $\mathbb{C}$-algebra are chosen, as well as generators:

$$\langle f_1, ..., f_m \rangle = \ker(\phi)$$

for the kernel ideal, then we obtain a bijection:

$$\text{mspec}(A) \leftrightarrow \{\text{maximal ideals } m_p \in \mathbb{C}[x_1, ..., x_m] \text{ containing } \langle f_1, ..., f_m \rangle\}$$

and the latter is in a natural bijection with the set of **points**:

$$V(f_1, ..., f_m) = \{p \in \mathbb{C}^n \mid f_1(p) = f_2(p) = \cdots = f_m(p) = 0\} \subset \mathbb{C}^n$$

i.e. with the set of solutions to the system of polynomial equations.

Looking back at the algebraic groups, we get natural bijections:

(a) $\text{mspec}(\mathbb{C}[\text{SL}(n, \mathbb{C})]) \leftrightarrow \text{SL}(n, \mathbb{C})$

(c) $\text{mspec}(\mathbb{C}[O(n, \mathbb{C})]) \leftrightarrow O(n, \mathbb{C})$

(d) $\text{mspec}(\mathbb{C}[\text{SO}(n, \mathbb{C})]) \leftrightarrow \text{SO}(n, \mathbb{C})$

and finally, (b),

$$\text{mspec}(\mathbb{C}[\text{GL}(n, \mathbb{C})]) \leftrightarrow \{(A, y) \mid \det(A)y = 1\} \subset \mathbb{C}^{n^2+1}$$

which **projects** bijectively onto the set $\text{GL}(n, \mathbb{C}) = \{A \mid \det(A) \neq 0\}$.

The Zariski Topology on $\text{mspec}(A)$ is also intrinsic. In this topology, ideals determine subsets of $\text{mspec}(A)$ via:

$$Z(I) = \{m \mid I \subset m\} \subset \text{mspec}(A)$$

which are, by definition, the **closed sets** in the Zariski topology.

Exercise 9.2. Check that this defines a topology on $\text{mspec}(A)$, if we throw in the empty set as a closed set.

Example 9.3.. This is a strange topology! For example, if $A = \mathbb{C}[x[$, then **every** ideal is principal, generated by $f(x) \in \mathbb{C}[x]$, so:

$$I = \langle f(x) \rangle = \langle c(x - p_1) \cdots (x - p_m) \rangle \Rightarrow Z(I) = \{p_1, ..., p_m\}$$

and the closed sets are the finite sets. This is very non-Hausdorff.

Exercise 9.3. The complements of *hypersurfaces*:

$$U_f := \text{mspec}(A) - Z_{\langle f \rangle} \text{ for } f \in A$$

are a basis for the Zariski topology in the strong sense that every nonempty open set is a **finite** union of basis open sets.

From the basis theorem, it follows that any decreasing chain:

$$Z(I_1) \supseteq Z(I_2) \supseteq \cdots$$

of closed subsets in the Zariski topology eventually stabilizes at:

$$Z(I_n) = Z(I_{n+1}) = \cdots$$

A topological space with this property is called *Noetherian*.

We want to consider the sheaf of regular functions on $X = \text{mspec}(A)$. For this it is useful to make the following:

Restriction. Suppose A is a domain (and finitely generated $\mathbb{C}$-algebra). Let $X = \text{mspec}(A)$ with the Zariski topology. Then:

(a) X is an *irreducible* Noetherian topological space: if $X = Z_1 \cup Z_2$ is a union of closed sets, then either $Z_1 = X$ or $Z_2 = X$.

(b) The fraction field $\mathbb{C}(A)$ of A defines the *rational functions* on X.

(c) An element $f \in A$ is evaluated at a point $x \in X$ by:

$$f(x) := f \pmod{m_x} \in \mathbb{C}$$

where m_x is the maximal ideal corresponding to x.

(d) Every element $\phi \in \mathbb{C}(A)$ has an open *domain of definition*

$$U_\phi = \{x \in X \mid \phi = \frac{f}{g} \text{ with } g(x) \neq 0\}$$

(e) A *sheaf* of regular functions $\mathcal{O}_X$ is defined by setting:

$$\mathcal{O}_X(U) = \{\phi \in \mathbb{C}(X) \mid U \subset U_\phi\}$$

Proposition 9.2. The global regular functions $\mathcal{O}_X(X)$ are all in A.

Proof. Clearly $A \subset \mathcal{O}_X(X)$. Conversely, if $\phi \in \mathcal{O}_X(X) \subset \mathbb{C}(X)$, consider the ideal $I = \{g \mid g\phi \in A\}$ of denominators of ϕ. For all $x \in X$, there is a $g \in I$ such that $g \notin m_x$. Thus, I is not in **any** maximal ideal of A (Nullstellensatz!), so $1 \in I$ and $\phi \in A$.

Remark. As a variation on Proposition 9.2, one can show:

$$\mathcal{O}_X(U_f) = A[f^{-1}]$$

if $f \in A$ is any non-zero element of A.

Example 9.4. Consider the example of $A = \mathbb{C}[x, y]$. Then:

$$X = \mathbb{C}^2, \ Z(\langle f \rangle) = \{p \in \mathbb{C}^2 \mid f(p) = 0\}$$

are plane curves, the complements of which satisfy:

$$\mathcal{O}_{\mathbb{C}^2}(U_f) = \mathbb{C}[x, y, f^{-1}]$$

but one can easily show that $\mathcal{O}_{\mathbb{C}^2}(\mathbb{C}^2 - \{0\}) = \mathbb{C}[x, y]$, so that somehow the regular functions on $\mathbb{C}^2 - \{0\}$ do not "see" the missing point.

Definition 9.6. The data consisting of:

(a) An irreducible Noetherian topological space $X = \text{mspec}(A)$

(b) The sheaf of regular functions $\mathcal{O}_X(U)$ with $\mathcal{O}_X(X) = A$

is the **affine variety** (of finite type over $\mathbb{C}$) associated to A.

Definition 9.7. A morphism of affine varieties (over $\mathbb{C}$) is:

(i) A continuous map $f : X \to Y$ (Zariski topologies!) such that:

(ii) For all $U \subset Y$, the function pull-back $f^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ maps regular functions to regular functions.

Remark. The global pull-back $f^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is a $\mathbb{C}$-algebra homomorphism $h : B \to A$ of the underlying domains, and *conversely,* a homomorphism $h : B \to A$ of $\mathbb{C}$-algebras determines $f : X \to Y$ defined by $f(m_x) = h^{-1}(m_x)$, as well as the map on sheaves.

Moment of Zen: The category of affine varieties $(X, \mathcal{O}_X)$ of finite type over $\mathbb{C}$ is equivalent (contravariantly, i.e. with arrows reversed) to the category of finitely generated $\mathbb{C}$-algebra domains A.

Given this equivalence, it is natural to ask of affine varieties:

Q1. What morphisms are associated to *surjective* maps of domains?

Q2. What morphisms are associated to *injective* maps of domains?

The first question is rather easy, because:

A surjective map of domains *factors* through the quotient:

$$h : B \to B/\ker(h) = A$$

where $P = \ker(h)$ is the prime ideal *kernel* of the homomorphism h.

When we reverse arrows, this is an *isomorphism* of affine varieties between $\text{mspec}(A) = X$ and the *closed subvariety* $\text{mspec}(B/P) \subset Y$ which is "an affine variety" structure on the irreducible closed set $Z(P)$. Such a map of affine varieties is called a **closed embedding**.

As an example of the second question, consider the inclusion:

$$B \subset A = B[g^{-1}] \text{ for some non-unit } g \in A$$

Then the arrrow-reversed map of affine varieties:

$$f : \text{mspec}(B[g^{-1}]) \to \ \text{mspec}(B) = Y$$

is an isomorphism from X to the *basic open set* $U_g \subset Y$. This map is called an **open embedding**.

But for a *general* inclusion of rings, the answer is more interesting!

Example 9.4. (a) Projections. Let $B \subset A = B[x_1, ..., x_n]$. Then:

$$f : \text{mspec}(A) \to \text{mspec}(B) = Y$$

is the **projection** $p : Y \times \mathbb{C}^n \to Y$.

(b) Blow-downs. Let $h : \mathbb{C}[x, xy] \subset \mathbb{C}[x, y]$ be the inclusion of one polynomial ring in another. Then:

$$f : \mathbb{C}^2 \to \mathbb{C}^2 \text{ is the map } f(a, b) = (a, ab)$$

whose image is neither open nor closed. It is the complement $U_y \subset \mathbb{C}^2$ of the y-axis together with the origin. Notice that $f^{-1}(0, 0) = \{(0, b)\}$ while $f^{-1}(a, c) = \{(a/c, c)\}$ is a single point otherwise. For this reason, we say that f **blows down** the y-axis onto the origin.

We want to define an abstract variety (of finite type over $\mathbb{C}$) to be an irreducible topological space with a sheaf of regular functions on it that is locally affine and globally Hausdorff, as we did for manifolds, but we have already mentioned a problem with the Hausdorff property for the Zariski topology. We need a categorical work-around.

Products exist in the category of affine varieties. The contravariant functor above tells us this is equivalent to the existence of *coproducts* in the category of $\mathbb{C}$-algebra domains. These are furnished by the tensor product of $\mathbb{C}$-algebras over $\mathbb{C}$:

$$A \otimes_{\mathbb{C}} B \text{ together with } a : A \otimes 1 \to A \otimes B, b : 1 \otimes B \to A \otimes B$$

is a coproduct (the only hard thing is to prove that it is a domain!) Via our contravariant functor, this means that:

$$\text{mspec}(A \otimes B), \text{with the maps } a^* \text{ and } b^* \text{ is a product!}$$

but this product of varieties does **not** have the product topology! For example, $\mathbb{C}^2 = \mathbb{C}^1 \times \mathbb{C}^1$ via this construction, but its Zariski topology has many more closed sets in it (plane curves!) than does the product of the coifinite topologies.

Observation. The diagonal mapping:

$$X \to X \times X$$

is the closed embedding corresponding to the surjective map:

$$A \otimes A \to A;\ a_1 \otimes a_2 \mapsto a_1 \cdot a_2$$

converting tensor product to ordinary product. The kernel is:

$$\langle a_i \otimes 1 - 1 \otimes a_i \mid a_i \in A \text{ generate } A \text{ as an algebra!}\rangle$$

which in particular proves that the diagonal in $X \times X$ is **closed**.

Looking back at the section on topology, we noticed (Exercise 6.2 (b)) that Hausdorff was equivalent to the diagonal being closed *provided that products had the product topology.* Here they don't, so we go with the categorical definition instead.

Definition 9.8. An object X in a category of topological spaces with products is *separated* if the diagonal $X \to X \times X$ is a homeomorphism onto a closed subset $\Delta \subset X \times X$.

Remark. Here the product doesn't have the product topology as a topological space. Amusingly, in Grothendieck's theory of schemes, the product of schemes isn't even the Cartesian product *as a set*!

By analogy with manifolds, we now try to make the following:

Definition 9.9. A *variety* is a Noetherian topological space X together with a sheaf of regular functions $\mathcal{O}_X$ so that the pair $(X, \mathcal{O}_X)$ is locally affine and globally separated.

But there is a problem here! What is the product of X with itself? We did it for affine varieties by appealing to the tensor product, but we have enlarged our category to include pairs $(X, \mathcal{O}_X)$ that are *locally affine*. Well, it can be done, even in Grothendieck's wild category of locally affine schemes. I refer you to Hartshorne's book or you can think about it as an ambitious exercise.

Exercises.

9.1. (a) A $\mathbb{C}$-algebra with no proper ideal other than 0 is a field.

(b) The quotient of a $\mathbb{C}$-algebra by a maximal ideal is a field.

9.2. (a) Check that declaring $Z(I)$ to be closed if $I \subset A$ is an ideal defines a topology on $\text{mspec}(A)$ (after you throw in the empty set). This is the Zariski topology.

(b) Check that $I(Z(I))$ contains the *radical* of I, defined by:

$$\sqrt{I} = \{f \in A \mid f^r \in I \text{ for some } r > 0\}$$

and also check that $\sqrt{I}$ is, in fact, an ideal.

Remark. It is a consequence of the Nullstellensatz that $\sqrt{I} = I(Z(I))$.

9.3. Check that the complements of *hypersurfaces*:

$$U_f := \text{mspec}(A) - Z_{\langle f \rangle} \text{ for } f \in A$$

are a basis for the Zariski topology in the strong sense that every nonempty open set is a **finite** union of basis open sets.

9.4. (Ambitious) Create a category of *locally affine spaces with sheaves* and prove that products exist in your category, so that the notion of being separated **makes sense**!