Moduli in Algebraic Geometry: An Introduction

Math 7800, Spring 2022. Instructor: Aaron Bertram

0.2. Schemes. Schemes are well-suited to the study of moduli problems through their **functor of points**. The points (in the usual sense) of a scheme X are prime ideals in a commutative ring, but the functor of points is the contravariant functor:

$$h_X : \mathit{Schemes} \to \mathit{Sets}$$

from the category of schemes to the category of sets defined on objects by:

$$h_X(S) = \mathrm{Mor}(S, X) = \{\mu : S \to X\}$$

and on morphisms by:

$$h_X(a : S \to T) = (a^* : h_X(T) \to h_X(S))$$

where a^* is the pre-composition of morphisms: $a^*(\nu : T \to X) = (\nu \circ a : S \to X)$.

This may seem like a curious definition, until you think about the motivation coming from the algebraic geometry of varieties defined by polynomials with coefficients in an arbitrary field k. Grothendieck's theory of schemes arose in part from the difficulty the Bourbaki school had with finding a definition of a variety in this context that captured all solutions to the equations in all field extensions of k.

The **spectrum** of a field $\mathrm{Spec}(k)$ is a one-point scheme equipped with (constant) functions with values in k. A scheme X defined by a system of polynomial equations is then a scheme **over** $\mathrm{Spec}(k)$, i.e. X is equipped with a morphism:

$$\pi : X \to \mathrm{Spec}(k)$$

if the *coefficients* of the polynomials defining X are in k. On the other hand, the *solution* set to the equations in a field extension K of k is:

$$X(K) := X(\mathrm{Spec}(K))$$

i.e. the set of maps from $\mathrm{Spec}(K)$ to X. For example, the scheme associated to the equation $x^2 + y^2 = 3$ is defined over $\mathbb{Q}$ but $X(\mathbb{Q}) = \emptyset$, while $X(\mathbb{R})$ is the set of points of the circle and $X(\mathbb{C})$ is the Riemann sphere minus two points.

The functor of points h_X is functorial in two senses. First, we have:

$$X(a \circ b) = X(b) \circ X(a) \text{ for } b : S \to T \text{ and } a : T \to U$$

establishing that h_X is a functor, but also, given a morphism $f : X \to Y$, we have:

$$f_* : h_X \to h_Y$$

a *natural transformation of functors* of points with:

$$f_* : h_X(S) \to h_Y(S) \text{ defined by } f_*(\mu) = f \circ \mu : S \to Y$$

i.e. post-composition with f, and the map on arrows:

$$f_*(h_X(a))(\nu) = f_*(a^*)(\nu) = a \circ \nu \circ f = a^*(f_*\nu) \in h_Y(a)$$

by the associativity of composition.

Yoneda's Lemma. Let $\mathcal{S}ch\mathcal{S}ets$ be the category of contravariant functors from schemes to sets (the objects) with natural transformations (the morphisms). Then $h_\bullet$ defined by $h_\bullet(X) = h_X$ and $h_\bullet(f) = f_*$ is a fully faithful functor. In particular, a scheme X is determined (up to isomorphism) by its functor of points.

Notice that this is in contrast with the traditional points of a scheme, which do not determine the scheme. For example, any two schemes that share the same reduced scheme structure have identical traditional points.

Definition. A contravariant functor h from schemes to sets is **representable** if h is isomorphic to h_X for some (unique up to isomorphism) scheme X.

This lemma applies to any category. The reason why the category of schemes is well-suited to invoking Yoneda's Lemma is that moduli problems in algebraic geometry set up well thanks to the existence of *fiber products* of schemes and *flat families* of coherent sheaves.

Affine Schemes are locally ringed spaces Spec(A) for a commutative ring A with 1 consisting of a set of (traditional) points; the prime ideals $\mathcal{P} \subset A$ with $1 \notin \mathcal{P}$ together with the Zariski topology with a basis of neighborhoods:

$$U_f = \{\mathcal{P} \mid f \notin \mathcal{P}\} \text{ for } f \in A$$

and a sheaf of communtative rings $\mathcal{O}_{\mathrm{Spec}(A)}$ on Spec(A), the **sheaf of regular functions**, determined by the localizations:

$$\mathcal{O}_{\mathrm{Spec}(A)}(U_f) = A_f$$

(at the multiplicative set $\{1, f, f^2, ..., \}$). The stalks $\mathcal{O}_{\mathrm{Spec}(A),\mathcal{P}} \cong A_{\mathcal{P}}$ of germs of regular functions are local rings whose maximal ideals obtained by localizing at the multiplicative set $A - \mathcal{P}$. A morphism of locally ringed spaces is a continuous map that pulls back regular functions to regular functions and elements of the maximal ideal to the maximal ideal. Then the category of commutative rings embeds contravariantly and fully faithfully into the category of locally ringed spaces.

An affine scheme Spec(A) is of finite type over an algebraically closed field k if A is finitely-generated as an algebra over k, in which case:

(i) The maximal ideals of A are in bijection with the points of a finite union of closed subvarieties X of affine space $\mathbb{A}^n_k$ for some n.

(ii) The set of (traditional) points of A are in bijection with the closed irreducible subsets of X in the Zariski topology (the closed subvarieties of X).

(iii) The regular functions on Spec(A) are the regular functions on X augmented by nilpotent elements ("regular functions" that evaluate to zero at all points!). The underlying reduced scheme is obtained by setting these nilpotents to zero.

Examples. Two basic examples of particular interest in moduli theory are:

(a) The schemes Spec(A) of finite-type over a field k, where A is an Artinian local ring, finitely generated as an algebra, and also as a vector space over k. E.g.

$$\mathrm{Spec}(k[x])/\langle x^2 \rangle$$

the "ring of dual numbers."

(b) The two-point schemes Spec(A) where A is a Discrete Valuation Ring (DVR) with fraction field K, maximal ideal $m = \langle \pi \rangle$ and residue field $k = A/m$. E.g.

$$\mathrm{Spec}(k[x]_{\langle x \rangle})$$

(or the stalk of the sheaf of regular functions at any non-singular point of a curve over Spec(k)). In this case $k \subset A$, but A is not finitely generated as a k-algebra.

Projective Schemes (of finite type over k) are locally ringed spaces $\mathrm{Proj}(R_\bullet)$ associated to finitely generated graded rings $R_\bullet$ over $k = R_0$, which we will assume for simplicity are generated in degree one, so that:

$$S = k[x_0, ..., x_n]_\bullet \to R_\bullet$$

is a surjective graded ring homomorphism, with choice of basis $x_0, ..., x_n \in R_1$. This data determines the projective scheme $\mathrm{Proj}(R_\bullet)$ **along with** an embedding $\mathrm{Proj}(R_\bullet) \subset \mathbb{P}^n_k$ which, in light of the previous section, means that the given data must determine a **line bundle** (invertible sheaf) on $\mathrm{Proj}(R_\bullet)$. Indeed, the points of $\mathrm{Proj}(R_\bullet)$ correspond to homogenous prime ideals $P_\bullet$ with $P_1 \neq R_1$, the Zariski topology has a basis of open sets U_F for homogeneous polynomials $F \in R_d$ defined as in the affine case, with sheaf of regular functions defined by letting $\mathcal{O}_{\mathrm{Proj}(R_\bullet)}(U_F)$ be the degree zero part of the localization R_F of $R_\bullet$ at the multiplicative set given by the powers of F. Each open subset U_F is isomorphic (as a locally ringed space) to $\mathrm{Spec}(R_F)$, which shows that $\mathrm{Proj}(R_\bullet)$ has a (finite) open cover by affine schemes. The invertible sheaf defining the embedding to projective space is given by the transition functions x_i/x_j on the overlaps $U_{x_i} \cap U_{x_j}$.

As in the affine case, the points of a projective scheme (of finite type) over an algebraically closed field correspond to the closed subvarieties in a finite union X of closed varieties in $\mathbb{P}^n$, augmented with nilpotent regular functions.

A locally ringed space X with sheaf of regular functions $\mathcal{O}_X$ is *locally affine* if it has an open cover by affine schemes, and Noetherian if the open cover by affine schemes $\mathrm{Spec}(A_i)$ may be chosen to be finite and the rings A_i Noetherian. Such schemes satisfy Noetherian induction; every descending chain of closed subschemes eventually stabilizes, where a closed subscheme Z is defined by a *sheaf of ideals* $\mathcal{I}_Z \subset \mathcal{O}_X$ with the property that $I_i = \mathcal{I}_Z(\mathrm{Spec}(A_i)) \subset A_i$ are compatible ideals (necessarily finitely generated) carving out a compatible set of closed subschemes of the affines in the cover. A morphism of schemes $Y \to X$ is a closed embedding if it factors through an isomorphism with a closed subscheme $Z \subset X$.

Fiber Products. Noetherian schemes *over a fixed scheme* have fiber products. That is, if $f : X \to S$ and $g : Y \to S$ are Noetherian S-schemes, then there is a Noetherian fiber product $X \times_S Y$ with projections:

$$\widetilde{f} : X \times_S Y \to Y \ \text{ and } \ \widetilde{g} : X \times_S Y \to X$$

that are uniquely determined morphisms of S-schemes (by the universal property).

Remark. When $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(C)$ are affine schemes, then $X \times_S Y = \mathrm{Spec}(B \otimes_A C)$, since the tensor product is the coproduct.

Base Extension. Via the fiber product we get the *base extension* (change of base) from S-schemes to T-schemes associated to a morphism $T \to S$ via the *fiber square*:

$$\begin{array}{ccc} X \times_S T & \to & X \\ \widetilde{f} \downarrow & & f \downarrow \\ T & \to & S \end{array}$$

Remark. The S-schemes are a category, with morphisms of S-schemes given by morphisms $X \to Y$ that commute with the given morphisms to S. In this context, base extension by a morphism $T \to S$ is a *functor* from the category of S-schemes to the category of T-schemes.

The fiber product is also used to define separated and proper **morphisms**.

(i) $f : X \to S$ is **separated** if the diagonal morphism:

$$\Delta : X \to X \times_S X$$

is a closed embedding of S-schemes.

(ii) $f : X \to S$ is **proper** if it is separated, of finite type, and universally closed.

Remarks. If $f : X \to S$ is a morphism of Noetherian schemes, then the preimage of every open affine subscheme $U = \mathrm{Spec}(A) \subset S$ is covered by finitely many affine schemes $V_i = \mathrm{Spec}(B_i) \subset f^{-1}(U)$ equipped with ring homomorphisms $A \to B_i$. The morphism is of finite type if each B_i is finitely generated as an A-algebra. Universally closed means that $f : X \to S$ *and every base extension* of f is closed as a mapping of topological spaces.

Examples. (a) Every morphism of affine schemes is separated.

(b) The affine line $\mathbb{A}^1_k$ is not proper, and any variety over k with a non-constant regular function is therefore also not proper.

(c) $f : X \to S$ is **projective** if it factors through a closed embedding: $X \hookrightarrow \mathbb{P}^n_S$ followed by the projection: $\mathbb{P}^n_S \to S$ where $\mathbb{P}^n_S$ is *projective n-space* over S. If S is a scheme defined over k, then $\mathbb{P}^n_S = \mathbb{P}^n_k \times_{\mathrm{Spec}(k)} S$ for projective space over k. More generally (i.e. in the "mixed characteristic" case) one needs to take the product with $\mathbb{P}^n_{\mathbb{Z}}$, the projective space over $\mathrm{Spec}(\mathbb{Z})$ (a final object in *Schemes*).

Theorem (Grothendieck) Projective morphisms are (separated and) proper.

Valuative Criteria. The separatedness and properness of finite-type morphisms $f : X \to S$ of Noetherian schemes can be checked by considering only the lifts of morphisms $\mathrm{Spec}(A) \to S$ as A varies over all DVRs. Recall that we let K be the field of fractions of A, and therefore there is an open inclusion $\mathrm{Spec}(K) \subset \mathrm{Spec}(A)$ (the open point) of schemes, corresponding to the inclusion $A \subset K$. Then f is:

(i) separated if each lift of $\mathrm{Spec}(K) \subset \mathrm{Spec}(A) \to S$ to a morphism $\mathrm{Spec}(K) \to X$ factors uniquely through a lift $\mathrm{Spec}(K) \subset \mathrm{Spec}(A) \to X$, if the latter lift exists.

(ii) proper if the latter lift exists whenever the former does.

These criteria allow us to assert that moduli problems themselves are separated (respectively proper), in the sense that if they are represented by a scheme, then that scheme is separated (respectively proper). This is checked by considering families over $\mathrm{Spec}(A)$ for DVRs A.

Not the Hilbert Scheme. Define a functor h on k-schemes S by:

(i) $h(S) = \{\text{closed embeddings } f_S : X \hookrightarrow \mathbb{P}^n_S \text{ of } S\text{-schemes}\}$

(ii) $h(a : T \to S) =$ base extension

The k-points $h(\mathrm{Spec}(k))$ are in bijection with the closed subschemes $X \subset \mathbb{P}^n_k$ as desired, leading one to believe that this might be a good functor to capture all closed subschemes of $\mathbb{P}^n_k$, but the *families* over other schemes S are poorly behaved. For example, if A is a DVR with $k \subset A$ isomorphic to the residue field, then the closed subscheme $X_A = \mathbb{P}^n_k \subset \mathbb{P}^n_{\mathrm{Spec(A)}}$ lives entirely over the closed point. That is, the base extension of X_A to the closed point is $X_k = \mathbb{P}^n_k$, but the base extension X_K to the open point is empty!

One could fix this by restricting to families X_S that are **smooth** over S, since this notion is preserved by base extension and indeed this functor is represented by a separated (and quasi-projective) scheme but its k-points only capture the non-singular projective subschemes $X_k \subset \mathbb{P}^n_k$. They do not, for example, capture the singular limits of families of non-singular subschemes. Instead, the notion of a flat family, discussed in the next section, captures all closed subschemes and is represented by a disjoint union (indexed by the Hilbert polynomial) of projective schemes over $\mathrm{Spec}(k)$. These are the Hilbert schemes.