1. (a) If $R$ is a UFD and $a, b \in R$ share no common prime factor, show that:

$$f : R/\langle a \rangle \oplus R/\langle b \rangle \rightarrow R/\langle ab \rangle; \quad f(x + aR, y + bR) = bx + ay + abR$$

is an injective ring homomorphism.

(b) Find an example of (a) in which $f$ is not surjective.
2. Let $S \subset R$ is a multiplicative subset of a commutative ring with 1,

(a) Given an $R$-module homomorphism $f : M \to N$, define the $S^{-1}R$-module homomorphism $S^{-1}f : S^{-1}M \to S^{-1}N$ in the only sensible way.

(b) If $f$ is surjective, show that $S^{-1}f$ is also surjective.

(c) If $M \subset N$, show that $S^{-1}N/S^{-1}M \cong S^{-1}(N/M)$. 
3. Find the invariant factor and primary decompositions of:
\[ \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/18\mathbb{Z} \]
4. Let $R = \mathbb{Q}[x]$ and consider the submodule $M \subset R^2$ generated by the elements $(x^2 - 1, x - 1)$ and $(x^2 + x, x)$. Write $M$ as a sum of cyclic modules.
5. Suppose $R = \mathbb{F}_3[x]$. Let $M$ be the $R$-module generated by $a, b, c \in M$ subject to the three relations:
- $-xa + x^2b + (x^2 - 1)c = 0$
- $xb + xc = 0$ and
- $xb + x^2c = 0$.

Find the invariant factor and primary decompositions of $M$. 
6. Put the following matrix in rational canonical and Jordan normal forms.

\[
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]
7. (a) Prove that $A : k^n \to k^n$ is diagonalizable (with diagonal entries in $k$) if and only if the minimal polynomial of $A$ has $n$ distinct roots in $k$.

(b) If some power of $A : \mathbb{C}^n \to \mathbb{C}^n$ is $I_n$, show that $A$ is diagonalizable.
8. Find all the matrices $A : k^4 \to k^4$ (up to similarity) with $A^5 = 0$. Do any of them satisfy $A^4 \neq 0$?