HW #4 – MATH 6310
FALL 2022

DUE: FRIDAY, OCTOBER 21

1. (a) If R is a UFD and $a, b \in R$ share no common prime factor, show that:

$$f : R/\langle a\rangle \oplus R/\langle b\rangle \to R/\langle ab\rangle;\ \ f(x + aR, y + bR) = bx + ay + abR$$

is an injective ring homomorphism.

(b) Find an example of (a) in which f is not surjective.

2. Let $S \subset R$ is a multiplicative subset of a commutative ring with 1,

(a) Given an R-module homomorphism $f : M \to N$, define the $S^{-1}R$-module homomorphism $S^{-1}f : S^{-1}M \to S^{-1}N$ in the only sensible way.

(b) If f is surjective, show that $S^{-1}f$ is also surjective.

(c) If $M \subset N$, show that $S^{-1}N/S^{-1}M \cong S^{-1}(N/M)$.

3. Find the invariant factor and primary decompositions of:

$$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/18\mathbb{Z}$$

4. Let $R = \mathbb{Q}[x]$ and consider the submodule $M \subset R^2$ generated by the elements $(x^2-1,\ x-1)$ and $(x^2+x,\ x)$. Write M as a sum of cyclic modules.

5. Suppose $R = \mathbb{F}_3[x]$. Let M be the R-module generated by $a, b, c \in M$ subject to the three relations:

- $-xa + x^2b + (x^2 - 1)c = 0$
- $xb + xc = 0$ and
- $xb + x^2c = 0$.

Find the invariant factor and primary decompositions of M.

6. Put the following matrix in rational canonical and Jordan normal forms.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

7. (a) Prove that $A : k^n \to k^n$ is diagonalizable (with diagonal entries in k) if and only if the minimal polynomial of A has n distinct roots in k.

(b) If some power of $A : \mathbb{C}^n \to \mathbb{C}^n$ is I_n, show that A is diagonalizable.

8. Find all the matrices $A : k^4 \to k^4$ (up to similarity) with $A^5 = 0$. Do any of them satisfy $A^4 \neq 0$?