HW #3 – MATH 6310
FALL 2022

DUE: FRIDAY, OCTOBER 7

Modified from exercises in Aluffi.

Let R be a commutative ring with 1.

1. Show that if the free R-modules R^m and R^n are isomorphic, then $m = n$. Thus the *rank* of a free module is well-defined.

2. Prove the second isomorphism theorem for R-modules:

(a) If S, T are submodules of an R-module M, show that both $S \cap T$ and

$$S + T = \{s + t \mid s \in S,\ t \in T\} \subset M$$

are submodules of M.

(b) Again, given S, T submodules of M, find an isomorphism:

$$f : \frac{S+T}{T} \to \frac{S}{S \cap T}$$

3. Prove the third isomorphism theorem. Given submodules

$$S \subset T \subset M$$

find an isomorphism:

$$f : \frac{(M/S)}{(T/S)} \to \frac{M}{T}$$

4. A nonzero R-module M is **simple** if its only submodules are $\{0\}$ and M.

(a) Find all the simple (finitely generated) $\mathbb{Z}$-modules.

(b) If k is a field, find all the simple $k[x]$-modules.

(c) If M and N are simple, prove that every R-module homomorphism $f : M \to N$ is 0 or else an isomorphism (this is *Schur's Lemma* for modules).

5. An R-module M is said to have *finite length* if there are submodules:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

with the property that each M_{i+1}/M_i is simple. Such a series of submodules is called a *composition series* for M.

(a) Prove that $\mathbb{Z}$ does not have finite length as a $\mathbb{Z}$-module but that each $\mathbb{Z}/d\mathbb{Z}$ does have finite length.

(b) Consider the $k[x]$-module $M = k[x]/\langle x^2 \rangle$.

Show that M has a composition series $0 = M_0 \subset M_1 \subset M_2 = M$ with

$$M_1 = k[x]/\langle x \rangle \text{ and } M_2/M_1 = k[x]/\langle x \rangle$$

but that M_2 is **not** isomorphic to $k[x]/\langle x \rangle \oplus k[x]/\langle x \rangle$.

6. (Challenging!) Prove that any two composition series for the same module M have the same length and have the the same simple "Jordan-Hölder" factors M_{i+1}/M_i (counted with multiplicity).

7. Suppose M is the cokernel of:

$$f : k[x]^2 \to k[x]^2 \text{ given by } f = \begin{bmatrix} x^2 - 1 & 0 \\ 0 & (x-1)^2 \end{bmatrix}$$

(a) Find M in the format of the Structure Theorem.

Hint: Your answer depends upon the characteristic of the field k!