HW #2 – MATH 6310
FALL 2022

DUE: FRIDAY, SEPTEMBER 9

Modified from exercises in Aluffi.

1. Prove the 2nd Iso Theorem. Let R be a commutative ring with 1.
If $S \subset R$ is a subring and $I \subset R$ is an ideal then
(a) $S + I \subset R$ is a subring and $I \subset S + I$ and $S \cap I \subset S$ are ideals and
(b) $S/(S \cap I)$ is isomorphic to $(S + I)/I$.

2. Let R be a domain. Prove that $R[x]$ and $R[[x]]$ are domains.

3. If $I, J \subset R$ are ideals in a commutative ring, let:

$$IJ = \{\sum_{i,j} s_i t_j \mid s_i \in I \text{ and } t_j \in J\}$$

be the finite sums of products of elements of I with elements of J.

(a) Prove that IJ is an ideal, and that $IJ \subset I \cap J$.

(b) Find an example where $IJ \neq I \cap J$.

4. In the context of Problem 3, prove that if $I + J = R$, then $IJ = I \cap J$.

5. (a) If k is an algebraically closed field, prove that every maximal ideal in $k[x]$ is of the form $\langle x - a \rangle$ for some $a \in k$.

(b) Find all the maximal ideals in $\mathbb{R}[x]$.

(c) Find polynomials $f(x)$ of every degree in $\mathbb{Q}[x]$ such that $\langle f(x) \rangle \subset \mathbb{Q}[x]$ is a maximal ideal.