HW #1 – MATH 6310
FALL 2022

DUE: FRIDAY, SEPTEMBER 2

Modified from exercises in Aluffi, Chapter III.

1. Let R be a commutative ring with 1. An element $x \in R$ is called *nilpotent* if $r^n = 0$ for some $n > 0$.

(a) Prove that the set of nilpotent elements of R is an ideal. This is the *nilradical* N of R.

(b) Show that R/N has no nonzero nilpotent elements. Rings with no nonzero nilpotents are called *reduced*.

2. (a) Prove that $x = \pm 1$ are the only solutions to $x^2 = 1$ in a field.

(b) Find a commutative ring where $x^2 = 1$ has more solutions.

3. Find all the commutative rings with $0, 1$ and exactly two other elements. Be explicit! Are there any non-commutative rings with four elements?

4. Let R be a commutative ring and consider the power series ring $R[[x]]$. Prove that $a_0 + a_1x + a_2x^2 + \dots$ has a multiplicative inverse if and only if a_0 has a multiplicative inverse in R.

5. The *center* of a (non-commutative) ring R consists of all elements a such that $ar = ra$ for all $r \in R$. Prove that the center is a subring of R and that the center of a division ring is a field. What is the center of the division ring of quaternions?

6. Let R be a ring containing $\mathbb{C}$ as a subring. Prove that there is no ring homomorphism $R \to \mathbb{R}$.