Riemann Surfaces and Graphs
4. Tropical Linear Series

In §1, an equivalence relation among divisors on graphs was defined via a series of chip-firings which were collected into a single effective divisor:

$$\sum_v c_v \cdot v \in V_d$$

If we instead think of this as a *function* $\phi : V \to \mathbb{Z};\ \phi(v) = c_v$ then chip-firings fit into the linear equivalence definition from §3 if we let:

$$\mathrm{ord}_v(\phi) = \sum_{\text{neighbors } w \text{ of } v} (\phi(w) - \phi(v)) \quad \text{and} \quad \mathrm{div}(\phi) = \sum_v \mathrm{ord}_v(\phi) \cdot v$$

because in that case $D + \mathrm{div}(\phi)$ is the divisor obtained from D by firing according to the divisor $\sum c_v \cdot v$. Indeed, with this definition,

$$\mathrm{div}(\phi + \psi) = \mathrm{div}(\phi) + \mathrm{div}(\psi)$$

and firing from $v \in V$ is $D \rightsquigarrow D + \mathrm{div}(1_v)$ for the indicator function of v.

In the context of meromorphic functions on Riemann surfaces, $\mathrm{div}(\phi)$ converted *multiplication* (of functions) to addition (of divisors), and sums of meromorphic functions satisfied the following property which in this context is played by the *maximum*:

Lemma 4.1. If $\phi, \psi : V \to \mathbb{Z}$ and $\mathrm{ord}_v(\phi) \ge d$ and $\mathrm{ord}_v(\psi) \ge d$, then:

$$\mathrm{ord}_v(\max\{\phi, \psi\}) \ge d$$

Proof. Without loss of generality, let $\phi(v) \ge \psi(v)$ and $\theta = \max\{\phi, \psi\}$. Then $\theta(w) - \theta(v) \ge \phi(w) - \phi(v)$ for all neighbors w of v. □

This allows us to both interpret $|D|$ as a *tropical* projective space, and extend the definition of equivalence under "chip firing" to *metric graphs*.

Definition 4.2. *Tropical arithmetic* on the real numbers is:

$$x \cdot_T y := x + y \text{ and } x +_T y := \max\{x, y\}$$

Exercise. With one exception, tropical arithmetic satisfies the properties of ordinary arithmetic, including the existence of the tropical additive identity when $\mathbb{R}$ is augmented to the *tropical numbers* $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$. The one exception is the non-existence of tropical additive identities of real numbers.

Definition 4.3. A tropical vector space is a set of vectors with a commuting addition and scalar multiplication by tropical numbers satisfying all the properties of a vector space except for the existence of additive inverses.

Example. (a) $\mathbb{T}^m$ is free m-space. It can be visualized as an infinite orthant.

(b) The *tropical span* of vectors $\vec{v}_1, ..., \vec{v}_n \in \mathbb{T}^m$ is the linear subspace:

$$\langle \vec{v}_1, \cdots \vec{v}_n \rangle := \left\{ \sum_T c_i \cdot_T \vec{v}_i := c_1 \cdot_T \vec{v}_1 +_T \cdots +_T c_n \cdot_T \vec{v}_n; \; c_i \in \mathbb{T} \right\}$$

That is, if $\vec{v}_i = (a_{i,1}, ..., a_{i,m})$, then:

$$\sum_T c_i \cdot_T v_i = \left(..., \max_i \{c_i + a_{i,j}\}, ... \right)$$

(c) Tropical linear subspaces of $\mathbb{T}^m$ do not behave as vector subspaces. There are tropical subspaces of $\mathbb{T}^3$ requiring uncountably many generators!

Exercise. Describe the tropical linear subspaces of $\mathbb{T}^3$ spanned by:

$$(i) \; (-1, 0, 0), (0, -1, 0) \text{ and } (1, 1, 0)$$

$$(ii) \; (1, 0, 0), (0, 1, 0) \text{ and } (-1, -1, 0)$$

Definition 4.4. If L is a tropical linear space, then:

$$\mathbb{P}(L) = \{L - \text{origin}\}/\mathbb{R}$$

where $\mathbb{R}$ acts on L by addition (= tropical scalar multiplication).

Examples. (0) $\mathbb{P}(\mathbb{T}^1)$ is a point.

(i) $\mathbb{P}(\mathbb{T}^2)$ is an infinite closed line segment.

(ii) $\mathbb{P}(\mathbb{T}^3)$ is an infinite triangle.

(iii) $\mathbb{P}(\mathbb{T}^m)$ is an infinite simplex.

As is the case with $\mathbb{CP}^n$, we can coordinatize $\mathbb{TP}^n = \mathbb{P}(\mathbb{T}^{n+1})$ with:

$$(c_0 : .... : c_n) \sim (c_0 + \lambda : \cdots : c_n + \lambda) \text{ for } \lambda \in \mathbb{R}$$

and realize $\mathbb{TP}^n$ as a union of subsets $U_i = \mathbb{T}^n$ or else as $U_0 \cup \mathbb{TP}^{n-1}$, the "points at infinity." From this point of view, the tropical projective plane is the infinite quadrant together with an infinite line segment interpolating between $(-\infty, +\infty)$ and $(+\infty, -\infty)$.

Examples. (i) The linear subspaces of $\mathbb{TP}^1$ are points and closed intervals.

(ii) The one-dimensional linear subspaces of $\mathbb{TP}^2$ are coordinate axes or else they have a "center" from which three segments emanate orthogonal to the coordinate axes.

Definition 4.5. A tropical polynomial $f(x)$ in one variable is:

$$f(x) = c_0 +_T c_1 \cdot_T x +_T \cdots +_T c_d \cdot_T x^d$$
$$= \max\{c_0, c_1 + x, \ldots, c_d + dx\}$$

and a tropical rational function is $\phi(x) = f(x) - g(x)$ (ordinary subtraction), which extends to a function $\phi : \mathbb{TP}^1 \to \mathbb{TP}^1$.

Definition 4.6. The graph of a tropical polynomial (or rational function) is the piecewise linear (integer slope) graph of the function $\phi : \mathbb{TP}^1 \to \mathbb{TP}^1$.

Definition 4.7. Given ϕ, let:

$$\mathrm{ord}_t(\phi) = \sum \text{outward slopes of } \phi \text{ at } t$$

(this is zero outside of the finitely many break points of the graph) and

$$\mathrm{div}(\phi) = \sum_{t \in \mathbb{TP}^1} \mathrm{ord}_t(\phi) \cdot t$$

is divisor of degree zero.

Remark. Given any divisor $D = \sum d_i \cdot t_i \in \mathbb{Z}[\mathbb{TP}^1]_0$, the rational functions:

$$\phi(t) = \lambda \cdot_T \prod_T (x +_T t_i)^{d_i} = \lambda + \sum d_i \max\{x, t_i\}$$

for $\lambda \in \mathbb{R}$ all satisfy $\mathrm{div}(\phi) = D$. This is a version of the fundamental theorem of algebra for the tropical numbers.

With this reformulation, we may generalize the chip-firing equivalence relation to a relation on divisors on metric graphs.

Definition 4.8. A *metric graph* Γ is a finite combinatorial graph together with the extra data:

$$\delta : E \to \mathbb{R}^{>0}$$

making Γ into a *metric space* in which the edge e has (linear) length $\delta(e)$ via the shortest path metric. The topology induced on Γ by this metric is a refinement of the coarse topology in §0 on the combinatorial graph.

A *divisor* on Γ is an element of $\mathbb{Z}[\Gamma]$. Elements of Γ_d are effective divisors.

Convention. A combinatorial graph Γ is generally given the structure of a metric graph by setting $\delta(e) = 1$ for all edges e.

Enhancement. Infinite length half-edges may be added to a finite metric graph, which may be capped off with a vertex "at infinity" at the cost of introducing a point at an infinite distance from the rest of the graph. Notice that $\mathbb{TP}^1$ itself is of this form, with two vertices at infinity.

Definition 4.9. A tropical rational function on a metric (enhanced) graph Γ is a piecewise linear function $\phi : \Gamma \to \mathbb{R}$ (or $\mathbb{TP}^1$) with integer slopes, with:

$$\mathrm{ord}_x \phi = \sum \text{outward slopes of } \phi \text{ at } x \in \Gamma \text{ and } \mathrm{div}(\phi) = \sum_{x \in \Gamma} \mathrm{ord}_x(\phi) \cdot x$$

Notice that at non-vertices, there are always two outward slopes, and $\mathrm{div}(\phi)$ is a finite sum, but that for vertices $v \in \Gamma$, there are $val(v)$ such slopes. We now define linear equivalence $\sim$ and linear series $|D|$ for divisors on metric graphs as in §3, and we note that the linear series of a combinatorial graph is exactly the subset of divisors in $|D|$ for the associated metric graph with unit side lengths that are supported on the vertices of the graph.

The following Lemma generalizes Corollary 1.5.

Lemma 4.10. If Γ is a metric tree, then $p \sim q$ for all $p, q \in \Gamma$.

Proof. Let P be the unique path from p to q, and let ϕ be a linear function on P with slope -1 along the path from p in the directon of q. Complete this to a function on the tree Γ by extending by constant functions. Then $\mathrm{div}(\phi) = q - p$, and so $p \sim q$.

On the other hand, consider the circle S^1 (of unit circumference). Then a divisor $D = \sum d_i \theta_i \in \mathbb{Z}[S^1]_0$ satisfies $D = \mathrm{div}(\phi)$ for a piecewise linear function $\phi : S^1 \to \mathbb{R}$ with integer slopes if and only if:

$$\sum_{d_i} d_i \theta_i = 0 \text{ as a point of } S^1$$

(compare with Lemma 2.11).

In particular, we get the following generalization of Corollary 1.8.

Corollary 4.11. If Γ has a circuit, then $p \not\sim q$ for points of the circuit.

Exercise. (i) Identify the linear series $|D|$ of degree d on $\mathbb{TP}^1$ with $\mathbb{TP}^d$.

(ii) Analyze the linear series $|2\theta|$ and $|3\theta|$ for $\theta \in S^1$.

(iii) Prove that $(S^1)_2$ is a Möbius strip. Describe $(S^1)_3$.

Tropical Plane Curves