Riemann Surfaces and Graphs
2. Meromorphic Functions and Meromorphic Differentials

Let S be a closed Riemann surface of genus g. Here we explore:

- meromorphic functions $\phi(z)$ (in the local coordinate z) and
- meromorphic differentials $\omega = \psi(z)dz$ (in the local coordinate z)

Definition 2.1. A *holomorphic map* $f : S \to T$ of Riemann surfaces is a continuous map with the property that if $f(p) = q$, then in local coordinates z near p and w near q, $w = f(z)$ is a holomorphic function.

Example. A meromorphic function $\phi \in \mathbb{C}(S)$ defines a holomorphic map:

$$f : S \to \mathbb{CP}^1$$

sending the poles of S to the point at infinity. In the local coordinate around $0 \in \mathbb{CP}^1$, this is the definition of a meromorphic function, and in the local coordinate around $\infty \in \mathbb{CP}^1$, this follows from the fact that if ϕ has a pole at p, then $1/\phi$ is holomorphic at p (with a zero at p).

We can say lot about the "shape" of a holomorphic map of Riemann surfaces. Since every non-constant holomorphic function in a neighborhood of $p \in \mathbb{C}$ has the form:

$$f(z) - f(p) = (z-p)^e g(z) \text{ with } g(p) \neq 0$$

it follows that $f(z) - f(p)$ has an analytic eth root in a neighborhood of p, which we use as a new local coordinate in terms of which $f(z) - f(p) = z^e$. For nearby points q, $f(z) - f(q) = z^e - (f(q) - f(p)) = z^e - q^e = (z-q)g(z)$ has the value $e = 1$. We call the value e the *ramification index* of $f(z)$ at p and say that $f(z)$ is *unramified* at p if the ramification index at p is 1.

Lemma 2.2. The ramification index at $p \in S$ of a non-constant holomorphic map $f : S \to T$ of Riemann surfaces is well-defined and the map is surjective and unramified at all but finitely many points.

Proof. The ramification index of a composition $f \circ g$ of holomorphic functions is the product of ramification indices, and therefore an invertible function is unramified. It follows that the ramification index is *independent* of the choice of local coordinates at p and $f(p)$, and therefore well-defined. Since nearby points to a ramified point are unramified, it follows that there can be no accumulation point of ramified points, and therefore since S is compact, there can only be finitely many of them.

Finally, it follows from the local description that the *image* of f is open and compact (because f is continuous), hence also closed and since T is assumed to be connected, the map f is surjective.

Definition 2.3. Given $f : S \to T$, let $R \subset S$ be the finite set of ramification points and $B = f(R) \subset T$ be the finite set of *branch* points of f.

Lemma 2.4. A non-constant holomorphic map $f : S \to T$ restricts to a finite covering space over the complement $T - B$ of the branch points.

Proof. Since f is a continuous map of compact spaces, it is proper, from which it follows that the restriction of f to $f^{-1}(T - B)$ is also proper and contains only unramified points. Near an unramified point, f is a locally invertible analytic map, from which it follows that the restriction of f is a covering space. Moreover, by properness of the map it follows that $f^{-1}(q)$ is finite for each $q \in T - B$. Note, however, that f is not, in general, a covering space at all points of $S - R$, since the restriction of a proper map to an open subset of the domain is not, in general, proper!

Definition 2.5. The *degree* of the map f is the degree of the covering map.

Shape of the Map. Fix a base point $q_0 \in T - B$. Above q_0 the covering has d sheets (where d is the topological degree of the map). As one traverses a path in $T - B$ starting at q_0, one may track the sheets of the cover. At a *ramification point* $p \in S$ of index e, there are e sheets that come together. "Winding" once around the branch point $b = f(p)$ permutes the e sheets cyclically. This happens simultaneously at all ramification points lying above b, giving an element of the symmetric group on d sheets with cycle decomposition given by the ramification indices of the points of $f^{-1}(b)$.

Let $\phi \in \mathbb{C}(T)$ be non-constant, and let $p \in S$ be a point of ramification index e for the map f, with $f(p) = q$. In suitable local coordinates z and w near p and q, respectively, we have:

$$w = z^e$$

So if $\mathrm{ord}_q(\phi) = m$, meaning that near q, $\phi(w) = w^m g(w)$ with $g(0) \neq 0$, then near p, $\phi \circ f(z) = (z^e)^m g(z^e)$, so $\mathrm{ord}_p(\phi \circ f) = me$. Recall that:

$$\mathrm{div}(\phi) = \sum_q \mathrm{ord}_q(\phi) \cdot q \in \mathbb{Z}[T]_0$$

is the divisor of zeroes (and poles) of ϕ.

Definition 2.6. $f^* : \mathbb{C}(T) \to \mathbb{C}(S)$ is the *pullback* map of fields given by:

$$f^*(\phi) = \phi \circ f$$

Then by the above remark:

$$(*)\ \mathrm{div}(f^*\phi) = \sum_{q \in T} \sum_{p \in f^{-1}(q)} e_p \cdot \mathrm{ord}_q(\phi) \cdot p$$

Example. Let $\phi \in \mathbb{C}(S)$ and let $f : S \to \mathbb{CP}^1$ be the associated map. Then $f^*z = \phi$, and the zeroes of ϕ occur at the points of $f^{-1}(0)$ with multiplicities equal to the ramification indices, while the poles of ϕ occur at the points of $f^{-1}(\infty)$ also with multiplicities equal to the (negatives) of the ramification indices. Since the **sum** of the ramification indices is equal to the degree of the map f, this gives another proof that the degree of $\mathrm{div}(\phi)$ is zero.

Now let $\omega = \psi(w)dw$ be a meromorphic **differential** on T.

Definition 2.7. The pull-back on differentials is defined by:

$$f^*\omega = f^*\psi(z)df(z) = \psi(f(z))f'(z)dz$$

where z is a local coordinate in a neighborhood of p and w is a local coordinate in a neighborhood of q with $w = f(z)$ in local coordinates.

Exercise. Check that this is well-defined. (Hint: Chain rule.)

The Riemann-Hurwitz Formula. Let $\omega = \psi(w)dw$ be a (meromorphic) differential form on T. Then:

$$\mathrm{div}(f^*\omega) = \mathrm{div}(f^*\psi) + \sum_{p \in R} (e_p - 1) \cdot p$$

In particular, the *degree* of the differential forms satisfy:

$$\deg(f^*\omega) = d \cdot \deg(\omega) + \sum_{p \in R} (e_p - 1)$$

Proof. In local coordinates if $w = z^e$ and $\psi(w) = w^m g(w)$, then

$$f^*\psi(w)dw = \psi(z^e)dz^e = (z^e)^m g(z^e) e z^{e-1} dz$$

This gives the first formula! The degree formula follows from $(*)$, which lets us conclude that $\deg(f^*\psi) = d \cdot \deg(\psi)$.

Corollary 2.8. If ω is a differential on S, then $\deg(\omega) = 2g - 2$.

Proof. Once the Corollary is true for one differential, it is true for all. Let $\phi \in \mathbb{C}(S)$ be a non-constant meromorphic function and: $f : S \to \mathbb{CP}^1$ the associated map. Recall that:

$$dz = -\frac{1}{w^2}dw$$

is a meromorphic differential on $\mathbb{CP}^1$, of degree -2. But:

$$\deg(d\phi) = \deg(f^*dz) = d \cdot \deg(dz) + \sum_{p \in R}(e_p - 1) = -2d + \sum_{p \in R}(e_p - 1)$$

by the degree formula above. On the other hand, let $B \subset \mathbb{CP}^1$ be the branch locus of the map f and *triangulate* $\mathbb{CP}^1$ with vertices $B \cup C$ for some additional set C of vertices. Then by Euler's formula for the sphere $\mathbb{CP}^1$,

$$(\#B + \#C) - \#E + \#F = 2$$

if E and F are the edges and faces of the triangulation. This triangulation *lifts* to a triangulation of S, with d times as many edges and faces, d times as many vertices of C, and vertices of B, **except** for the fact that e_p vertices collapse to one at each ramification point p. Thus,

$$2d - \sum_p (e_p - 1) = 2 - 2g$$

by Euler's formula again, which completes the proof. □

We will use the following Hodge-theoretic result:

HT1. The *holomorphic differentials* on S are a g-dimensional vector space.

Note: A differential $\omega = \psi(z)dz$ is *holomorphc* if $\text{ord}_p(\psi) \geq 0$ for all $p \in S$.

Genus One. By HT1, there is one holomorphic differential ω (up to scalar multiples) on S which has **no zeroes** by Corollary 2.8. Choose a base point $p_0 \in S$ and, for paths in S starting from p_0, integrate the one-form ω along the path. If γ is a **loop**, then $\int_\gamma \omega \in \mathbb{C}$ only depends on the homology class of γ and we get a *period map*:

$$\rho : \mathrm{H}_1(S, \mathbb{Z}) \to \mathbb{C};\ [\gamma] \mapsto \int_\gamma \omega$$

a homomorphism of abelian groups, mapping $\mathrm{H}_1(S, \mathbb{Z})$ onto a *lattice* $\Lambda \subset \mathbb{C}$.

This in turn defines the holomorphic *Abel-Jacobi* map:

$$a : S \to \mathbb{C}/\Lambda;\ a(p) = \int_{p_0}^{p} \omega$$

which is well-defined since any two paths from p_0 to p differ by a loop! This map is unramified, with $\omega = a^*dz$, and we will see that it is an isomorphism. We may choose generators λ_1, λ_2 for Λ so that:

$$\mathrm{Im}(\lambda_2/\lambda_1) > 0$$

and let P be the *fundamental domain*; i.e. the parallelogram with vertices $0, \lambda_1, \lambda_1 + \lambda_2, \lambda_2$ whose boundary ∂P is oriented by $\mathbb{C}$.

Let $\phi \in \mathbb{C}(S)$ be a non-constant meromorphic function, interpreted as a doubly-periodic function on $\mathbb{C}$, i.e. $\phi(z + \lambda) = \phi(z)$ for all $\lambda \in \Lambda$. Then if

$$\phi(z) = c_{-d}(z-a)^{-d} + \cdots + c_{-1}(z-a)^{-1} + c_0 + \cdots$$

is the Laurent series expansion near $a \in \mathbb{C}$, let $\mathrm{res}_a(\phi) = c_{-1}$ and note:

$$\frac{1}{2\pi i} \int_{\partial P} \phi(z)dz = \sum_{a \in P} \mathrm{res}_a(\phi) = 0$$

assuming that ϕ has no poles on ∂P. If ϕ does have such poles, then replace P by a translate $P + z_0$ to get the same result:

Lemma 2.9. The sum of residues of a meromorphic function on S is zero.

Corollary 2.10. There is no $\phi \in \mathbb{C}(S)$ with a single simple pole.

Remark. There is a more direct way to see Corollary 2.10. Namely, such a meromorphic function would determine a holomorphic map $f : S \to \mathbb{CP}^1$, which is necessarily an isomorphism. But $\mathbb{CP}^1$ is a sphere and S is a torus.

Next, starting with an arbitrary $\phi \in \mathbb{C}(S)$, consider the integral:

$$\frac{1}{2\pi i} \int_{\partial P} z \cdot \frac{d\phi}{\phi}$$

On the one hand, by double periodicity this is:

$$\frac{1}{2\pi i} \left(\int_0^{\lambda_2} \lambda_1 \frac{d\phi}{\phi} - \int_0^{\lambda_1} \lambda_2 \frac{d\phi}{\phi} \right) = m\lambda_1 - n\lambda_2 \text{ for winding numbers } m, n \in \mathbb{Z}$$

On the other hand, the residue of the differential at $a \in \mathbb{C}$ is $a \cdot \mathrm{ord}_a(\phi)$ from which we conclue:

Lemma 2.11. For each $\phi \in \mathbb{C}(S)$,

$$\sum_{a \in P} a \cdot \mathrm{ord}_a(\phi) = 0 \in \mathbb{C}/\Lambda$$

where this sum is taken *in the group law* of $S = \mathbb{C}/\Lambda$.

Definition 2.12. The *Weierstrass* $\mathcal{P}$ function:

$$\mathcal{P}(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda - \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

defines a doubly-periodic meromorphic function on $\mathbb{C}$ hence a meromorphic function on S with a single pole (of multiplicity two) at $0 \in S$.

Expanding, we get:

$$\mathcal{P}(z) = z^{-2} + 2 \left(\sum_{\lambda \neq 0} \frac{1}{\lambda^3} \right) z + 3 \left(\sum_{\lambda \neq 0} \frac{1}{\lambda^4} \right) z^2 + \cdots$$

but $\mathcal{P}(z)$ is an **even** function, so we may write:

$$\mathcal{P}(z) = z^{-2} + 3G_2 z^2 + 5G_3 z^4 + \cdots \text{ and}$$
$$\mathcal{P}'(z) = -2z^{-3} + 6G_2 z + 20 G_3 z^3 + \cdots$$

letting

$$G_k = \sum_{\lambda \neq 0} \frac{1}{\lambda^{2k}}$$

A little algebra then gives an *algebraic* relation between $\mathcal{P}$ and $\mathcal{P}'$:

$$\phi(z) := \mathcal{P}'(z)^2 - 4\mathcal{P}(z)^3 + 60 G_2 \mathcal{P}(z) + 140 G_3$$

is a doubly-periodic *holomorphic* function with $\phi(0) = 0$. So $\phi = 0$.

Genus Two Let ω_1, ω_2 be linearly independent holomorphic differentials and consider the meromorphic function ϕ satisfying:

$$\phi \omega_1 = \omega_2$$

Since $\deg(\omega_i) = 2$, it follows that ϕ has two poles and two zeroes, and:

$$f : S \to \mathbb{CP}^1$$

has degree two, with 6 ramification points by the Riemann-Hurwitz formula:

$$2 = 2g - 2 = \deg(\omega_i) = -2(2) + \sum_{p \in R} (e_p - 1) = -4 + \#R$$