Algebraic Curves/Fall 2015

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7. Riemann-Roch. Let D be a divisor (not necesarily effective) on a non-singular curve $C \subset \mathbb{CP}^n$. Recall that:

$$L(D) = \{\phi \in K(C)^* \mid \operatorname{div}(\phi) + D \geq 0\} \cup \{0\} \subset K(C)$$

is a finite-dimensional vector space over $\mathbb{C}$ of dimension $l(D)$.

Theorem 7.1 ("Classical" Riemann-Roch).

$$l(D) - l(K_C - D) = \deg(D) + 1 - g$$

where g is *defined* by the equation $\deg(K_C) = 2g - 2$.

We will prove this with a mix of algebra and analysis, following Mumford's *Algebraic Geometry I; Complex Projective Varieties.*

A Plausibility Argument. A (rational) differential $\omega \in \Omega(C)$ has a well-defined notion of a *residue* at each point $p \in C$. If z is a local (analytic) coordinate near p, with $z = 0$ at p, and if:

$$\omega = (b_{-d}z^{-d} + \cdots + b_{-1}z^{-1} + b_0 + ...)dz$$

then

$$res_p(\omega) = \frac{1}{2\pi i}\int_\gamma \omega = b_{-1}$$

where γ is an oriented (small) loop around p. This is remarkable, since it tells us that the coefficient of z^{-1} is intrinsic to the differential, and does not depend upon the choice of analytic local coordinate.

If $D = \sum d_i p_i \geq 0$ and z_i are local coordinates near p_i, we may let:

$$V = \left\{a_{i,-d_i}z_i^{-d_i} + ... + a_{i,-1}z_i^{-1}\right\}_{i=1}^n$$

be the vector space of "potential Laurent parts" of a function $f \in L(D)$. The *Mittag-Loeffler problem* asks when a potential Laurent part **is** the collection of Laurent tails of some $f \in L(D)$. Notice that if f_1, f_2 both solve the same Mittag-Loeffler problem, then:

$$f_1 - f_2 \text{ is holomorphic everywhere, hence constant}$$

so the solutions are unique, up to addition of a constant.

With residues, we see that the **regular** differentials on C produce *conditions* on solvability of the Mittag-Loeffler problem. Specifically:

$$\sum_{i=1}^n res_{p_i}(f\omega) = 0$$

for any $f \in L(D)$ and $\omega \in \Omega[C]$ by Stokes' Theorem.

This is a linear condition on Laurent parts in V. If

$$\{(b_{i,0} + \cdots b_{i,d_i-1}z_i^{d_i-1})dz_i\}_{i=1}^n$$

are the initial parts of the differential ω at each p_i, then:

$$\sum_{i=1}^n res_{p_i}(f\omega) = \sum_{i=1}^n (a_{i,-d_i}b_{i,d_i-1} + ... + a_{i,-1}b_{i,0})$$

and this is only identically zero if all the initial parts of ω *are zero*, i.e. $\omega \in L(K_C)$ fails to impose a linear condition on V iff $\omega \in L(K_C - D)$. Thus:

$$\dim(L(D)/\mathbb{C}) \leq \dim(V) - \dim(L(K_C)/L(K_C - D))$$

which (since $\dim(V) = \deg(D)$) gives an *inequality*:

$$l(D) - l(K_C - D) \leq \deg(D) + 1 - l(K_C)$$

We will see that the inequality is an equality, and that $l(K_C) = g$ which will give us the Riemann-Roch Theorem.

A Reduction. Suppose for every D there is a divisor E such that:

(i) $E - D \geq 0$ (i.e. D "is contained in" E), and

(ii) the Riemann-Roch Theorem holds for E.

Then the Riemann-Roch Theorem holds for every D.

Proof. If $D = \sum d_i p_i$ and $E = \sum e_i p_i$ with $e_i \geq d_i$, let:

$$V = \{a_{i,-e_i}z_i^{-e_i} + \cdots + a_{i,-d_i+1}z_i^{-d_i+1}\}_{i=1}^n$$

be the space of Laurent tails "between" an $f \in L(D)$ and a $g \in L(E)$. Then the natural "Laurent tail map" T has kernel $L(D)$:

$$0 \to L(D) \to L(E) \xrightarrow{T} V$$

since a rational function in $L(D)$ is a rational function in $L(E)$ with no Laurent tail between D and E.

Next, consider the residue pairing with a differential $\omega \in L(K_C - D)$:

$$res : V \times L(K_C - D) \to \mathbb{C};\ v \times \omega \mapsto \sum_i res_{p_i}(v \cdot \omega)$$

where $v \cdot \omega$ is understood to be the set of n locally defined differentials:

$$\{a_{i,-e_i}z_i^{-e_i} + \cdots + a_{i,-d_i+1}z_i^{-d_i+1}\} \cdot \omega$$

near each point $p_i \in C$. Then $res(v, \omega) = 0\ \forall v \in V \Leftrightarrow\ \omega \in L(K_C - E)$ since only a differential with zeroes of order e_i or more at each p_i will produce a zero overall residue (as in the plausibility argument above).

But if $g \in L(E)$, then

$$\sum_{i=1}^{n} res_{p_i}(T(g) \cdot \omega) = \sum_{i=1}^{n} res_{p_i}(g\omega) = 0$$

by Stokes' theorem, so the image of T pairs with zero against any differential in $L(K_C - D)$. This gives us the following sequence of vector spaces that is exact everywhere except (possibly) the middle:

$$0 \to L(E)/L(D) \stackrel{T}{\to} V \stackrel{res}{\to} L(K_C - D)^*/L(K_C - E)^* \to 0$$

where the latter map *res* is defined via the residue pairing.

It follows that:

$$\deg(E) - \deg(D) = \dim(V) \geq l(E) - l(D) + l(K_C - D) - l(K_C - E)$$

and therefore that if:

$$l(E) - l(K_C - E) = \deg(E) + 1 - g$$

then:

$$l(D) - l(K_C - D) \geq \deg(D) + 1 - g$$

On the other hand, we may apply the same argument with the roles of D and $K_C - D$ reversed (containing $K_C - D$ in a divisor E for which Riemann-Roch holds) to get:

$$l(K_C - D) - l(D) \geq \deg(K_C - D) + 1 - g = (2g - 2) - \deg(D) + 1 - g$$

which gives us the opposite inequality and hence equality. □

Now we get to the heart of the matter by connecting linear series with the homogeneous coordinate ring:

$$R = \mathbb{C}[x_0, ..., x_n]/I(C) \text{ of the embedded curve } C \subset \mathbb{CP}^n$$

Observation. Each homogeneous $F_d \in R_d$ defines an effective divisor E_d on C via the following:

$$\mathrm{ord}_p(F_d) := \mathrm{ord}_p(F_d/G) \text{ for } \textbf{any } G \in R_d \text{ with } G(p) \neq 0 \text{ and}$$

$$E_d := \mathrm{div}(F_d) = \sum_{p \in C} \mathrm{ord}_p(F_d) \cdot p$$

and notice that if $\mathrm{div}(F'_d) = E'_d$, then $\mathrm{div}(F'_d/F_d) + E_d = E'_d$ so all such divisors are linearly equivalent. We get injective maps:

$$f_d : R_d \to L(E_d);\ G \mapsto G/F_d \text{ for all } d \geq 0$$

Proposition 7.1. There is a d_0 such that f_d is surjective for all $d \geq d_0$.

In other words, there is a d_0 such that:

(*) If $d \geq d_0$ and $D \in |E_d|$ then $D = \mathrm{div}(G)$ for some $G \in R_d$.

This will show, in particular, that $l(E_d) = \dim(R_d)$ for all $d \geq d_0$ is computed by the Hilbert polynomial of R.

Assume C does not lie in any of the coordinate hyperplanes, and let $H_i = \text{div}(x_i)$, and consider any of the exact sequences:

$$0 \to R_{d-1} \stackrel{\cdot x_i}{\to} R_d \stackrel{r_d}{\to} (R_{H_i})_d$$

Then:

$$R_{H_i} = \mathbb{C}[x_0, ..., x_n]/\langle x_i, I(C)\rangle$$

has constant Hilbert polynomial $\delta = \deg(H_i)$, the **degree** of $C \subset \mathbb{CP}^n$, so the Hilbert polynomial of R is $h_R(d) = d\delta + c$ for some constant c, and therefore (after possibly raising the value of d_0),

$$l(E_d) = d\delta + c = \deg(E_d) + c \text{ for all } d \geq d_0$$

since $\deg(E_d) = d \cdot \deg(H_i)$. Also, since $\deg(E_d) > 2g - 2$ for large d, if we can additionally show that

$$c = 1 - g$$

then we have the Riemann-Roch theorem for all $E = E_d$ and $d \geq d_0$.

To prove the Proposition, we will use three tools:

(a) **Noether Normalization.** Under a "general" projection:

$$\pi_V : C \to \mathbb{CP}^1$$

(from a codimension two subspace $V \subset \mathbb{C}^{n+1}$), R is a finitely generated graded module over the homogeneous coordinate ring $\mathbb{C}[z_0, z_1]$ of $\mathbb{CP}^1$ (with $z_0 = \sum a_i x_i$ and $z_1 = \sum b_i x_i$).

(b) **Nullstellensatz.** If $\phi \in K(C)$ is regular at all points of $C \cap U_i$, i.e. if $\phi \in \mathcal{O}_{C,p}$ for all $p \in C \cap U_i$, then:

$$\phi \in \mathbb{C}[C \cap U_i] = \left\{ \frac{F}{x_i^N} \mid F \in R_N \right\} \subset K(C)$$

is in the coordinate ring of the affine curve $C \cap U_i \subset U_i = \mathbb{C}^n$.

(c) Let M be a finitely generated graded torsion-free module over $\mathbb{C}[z_0, ..., z_r]$, and let M_K be the localization of M with respect to the field $\mathbb{C}(z_0, ..., z_r)$. Then there is a d_0 such that, for all $d \geq d_0$, if

$$m \in M_K \text{ and } z_0^N m, ..., z_r^N m \in M_{N+d} \text{ for some } N$$

then

$$m \in M_d$$

Let us assume (a)-(c) for now and use them to prove the Proposition. Suppose $D \in |E_d|$. Since $|E_d| = |dH_i|$ for each i, there are rational functions $\phi_i \in K(C)$ such that:

$$\mathrm{div}(\phi_i) = D - dH_i$$

from which we conclude that $\mathrm{div}(\phi_i/\phi_j) = \mathrm{div}(x_j^d/x_i^d)$ hence ϕ_i/ϕ_j is a constant multiple of x_j^d/x_i^d. After multiplying each ϕ_i by a suitable constant, we can arrange for:

$$F = \phi_0 x_0^d = \phi_1 x_1^d = \cdots = \phi_n x_n^d \in K(R)$$

and moreover $\mathrm{div}(F) = D$, so we need to show that there is a d_0 such that $F \in R_d$ if $d \geq d_0$. We now invoke (a)-(c) as follows:

(a) Choose a generic projection so that R is a finitely generated (graded) module over $\mathbb{C}[z_0, z_1]$. It follows that the field of fractions $K(R)$ agrees with the localization R_K of R at $K = \mathbb{C}(z_0, z_1)$, and in particular that if $m_1, ..., m_k$ generate R as a $\mathbb{C}[z_0, z_1]$-module, then a subset of the $m_i's$ are a basis for R_K as a vector space over $\mathbb{C}(z_0, z_1)$.

(b) Since $\mathrm{div}(\phi_i) = D - dH_i$ only has negative coefficients at points of $C - C \cap U_i$, it follows that the $\phi_i \in \mathcal{O}_{C,p}$ for all $p \in C \cap U_i$, hence $\phi_i \in \mathbb{C}[C \cap U_i]$, and so for some N and all i, we have $x_i^N \phi_i \in R_N$, so:

$$x_0^N F, ..., x_n^N F \in R_{N+d}$$

(c) It follows that for $N' > (n+1)(N-1)$, every monomial of degree N' in $x_0, ..., x_n$ has degree N or more in some x_i, hence:

$$z_0^{N'} F, z_1^{N'} F \in R_{N'+d}$$

(since $z_0 = \sum_{i=0}^n a_i x_i$ and $z_1 = \sum_{i=0}^n b_i x_i$). Thus there is a d_0 such that, for all $d \geq d_0$, we have $F \in R_d$, proving Proposition 7.1. □

Computation of the Constant. Consider a projection:

$$\pi_V : C \to \mathbb{CP}^2$$

for $V \subset \mathbb{C}^{n+1}$ a general subspace of codimension three.

Claim. The image of π_V is a nodal curve, and π_V "resolves" the nodes.

Sketch of the Proof. Any projection π_V is a composition:

$$\pi_{q_r} \circ \cdots \circ \pi_{q_1} : C \to \mathbb{CP}^2$$

of projections from points (in successively smaller projective spaces). So it suffices to show that for general choice of $q \in \mathbb{CP}^n$,

$$\pi_q : C \to \mathbb{CP}^{n-1}$$

is an embedding if $n > 3$ and a resolution of a nodal curve if $n = 3$.

This is accomplished with a *dimension count*. Namely, if $C \subset \mathbb{CP}^n$, consider the "secant" mapping to the Grassmannian:

$$s : C \times C \to Gr(2, \mathbb{C}^{n+1});\ (p_1, p_2) \mapsto \overline{p_1 p_2}$$

and the incidence correspondence:

$$\begin{array}{ccc} & Fl(1,2,\mathbb{C}^{n+1}) & \\ \swarrow \pi_1 & & \pi_2 \searrow \\ \mathbb{CP}^n & & Gr(2,\mathbb{C}^{n+1}) \end{array}$$

from the flag manifold. Then the locus of points $q \in \mathbb{CP}^n$ that lie on a secant line of C is:

$$\pi_1(\pi_2^{-1}(s(C \times C)))$$

which has dimension at most 3, and therefore cannot fill $\mathbb{CP}^n$ if $n > 3$ and the projection is injective if q lies on no secant line. Similarly, for the tangent map $t : C \to Gr(2, \mathbb{C}^{n+1})$:

$$\pi_1(\pi_2^{-1} t(C))$$

is (at most) two-dimensional, and it follows that the projection is an immersion if q lies on no tangent line. This shows that the general projection to $\mathbb{CP}^3$ (or higher) is an embedding, and moreover a general projection to $\mathbb{CP}^2$ is an immersion. To see that all the singular points of the latter are nodes requires a more delicate but similar analysis of the loci of trisecant lines and "parallel tangent" lines to a curve in $\mathbb{CP}^3$.

We are now ready to invoke the genus computation of §6. Namely,

$$g = \binom{\delta - 1}{2} - \nu$$

where $2g - 2 = \deg(K_C)$ and ν is the number of nodes in $\pi_V(C) \subset \mathbb{CP}^2$. Finally, we need a Hilbert polynomial computation:

$$h_{\pi_V(C)}(d) = \binom{d+2}{2} - \binom{d+2-\delta}{2} = d\delta + 1 - \binom{\delta - 1}{2}$$

because $\pi_V(C) \subset \mathbb{CP}^n$ is a plane curve of degree δ, and the Hilbert polynomial of $C \subset \mathbb{CP}^n$ can be shown to satisfy:

$$h_C(d) = h_{\pi_V(C)}(d) + \nu$$

giving the desired computation of the constant term.

Finally, Riemann-Roch follows from the reduction since every divisor D on C can be evidently contained in a divisor of the form E_d for any sufficiently large value of d! $\square$