

Algebraic Curves/Fall 2015

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1. Introduction. What is a complex curve?

(Geometry) It's a Riemann surface, that is, a compact oriented two-dimensional real manifold Σ with a complex structure. Each such Σ has a numerical invariant, namely the genus g , which is informally the number of holes, and may be computed by triangulating the surface and computing the topological Euler characteristic $\chi(\Sigma) = 2 - 2g$.

(Algebra) It's a complex projective manifold C of dimension one. As such, it has a field of rational functions $K(C)$ of transcendence degree one over the field \mathbb{C} of scalars. The curve can be recovered from the field as the set of discrete valuation rings of K/\mathbb{C} , which has the “natural” structure of a projective manifold. There is a *canonical* line bundle ω_C , whose degree captures the genus via $\deg(\omega_C) = 2g - 2$. Alternatively, the space of sections of this line bundle has dimension g , and the “algebraic” Euler characteristic of C is $1 - g$.

There are advantages to both points of view. Algebraically, we can similarly treat curves, possibly with singularities, over any field k whereas the geometric point of view lends itself to applying tools from topology and differential geometry. To see that they define the same thing will take some thought. My bias will be the algebraic viewpoint.

The first example is the Riemann sphere.

Definition 1.1. The *Riemann sphere* is the complex manifold:

$$\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$$

that compactifies the complex numbers by adding a point at infinity. This may be done by identifying \mathbb{C} with the “slopes” of non-vertical lines through the origin in \mathbb{C}^2 with the point at ∞ as the vertical line. Precisely, let:

$$\mathbb{CP}^1 = \{\text{lines through the origin in } \mathbb{C}^2 \text{ (with coordinates } x_0, x_1)\}$$

and define two subsets of \mathbb{CP}^1 by:

$$U_0 = \{\text{lines of the form } \lambda \cdot (1, y_1) \text{ for } \lambda \in \mathbb{C}\}$$

$$U_1 = \{\text{lines of the form } \lambda \cdot (y_0, 1) \text{ for } \lambda \in \mathbb{C}\}$$

Each is equal to \mathbb{CP}^1 minus one point, which is the origin of the other:

$$(\mathbb{C} =) U_0 = \mathbb{CP}^1 - \text{the line } \lambda \cdot (0, 1) \in U_1$$

$$(\mathbb{C} =) U_1 = \mathbb{CP}^1 - \text{the line } \lambda \cdot (1, 0) \in U_0$$

and on the overlap $U_0 \cap U_1$, the y -coordinates are transformed via:

$$\lambda \cdot (1, y_1) = \lambda \cdot (y_1^{-1}, 1) \text{ and } \lambda \cdot (y_0, 1) = \lambda \cdot (1, y_0^{-1})$$

Thus two copies of \mathbb{C} are glued along the complements of the origin in each by the inverse map $y \mapsto y^{-1}$ to obtain \mathbb{CP}^1 . But this realization of \mathbb{CP}^1 as lines through the origin allows for the linear action of a group:

Definition 1.2. The *projective linear group* $\text{PGL}(2, \mathbb{C})$ is the quotient:

$$\text{GL}(2, \mathbb{C})/\mathbb{C}^*$$

of the group of invertible 2×2 matrices by the multiples of the identity.

The standard action of $\text{GL}(2, \mathbb{C})$ results in a well-defined action:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} ax_0 + bx_1 \\ cx_0 + dx_1 \end{bmatrix}$$

of $\text{PGL}(2, \mathbb{C})$ on the lines through the origin in \mathbb{C}^2 , i.e. points of \mathbb{CP}^1 . It is convenient to write a line in *homogeneous coordinates*:

$$(x_0 : x_1) = \lambda \cdot (x_0, x_1)$$

and then the action above takes $(x_0 : x_1)$ to $(ax_0 + bx_1 : cx_0 + dx_1)$. Notice that in the coordinate y_0 , this action has the following form:

$$A \cdot (y_0 : 1) = \left(\frac{ay_0 + b}{cy_0 + d} : 1 \right)$$

and is thus only well-defined as a map from \mathbb{C} to itself away from the value $y = -d/c$. This “rational map” is known as a *linear fractional transformation* of \mathbb{C} .

This action of $\text{PGL}(2, \mathbb{C})$ on \mathbb{CP}^1 is transitive. In fact:

Exercise 1.1. (a) Any three distinct points:

$$(p_1 : q_1), (p_2 : q_2), (p_3 : q_3) \in \mathbb{CP}^1$$

may be mapped, in order, to the points:

$$0 = (0 : 1), 1 = (1 : 1) \text{ and } \infty = (1 : 0), \text{ respectively } \in \mathbb{CP}^1$$

by a uniquely determined element $A \in \text{PGL}(2, \mathbb{C})$. Find this matrix, and apply it to an arbitrary fourth point $(p_4 : q_4)$. This is the *cross ratio* of the four ordered points.

(b) What happens to the cross ratio when the points are permuted?

For an exercise with a more geometric flavor,

Exercise 1.2. What happens to the (real) lines and circles in \mathbb{C} when $\mathbb{C} = U_1$ is acted upon by a linear fractional transformation?

We may generalize the construction of the Riemann sphere to get:

Definition 1.2. *Complex projective space* \mathbb{CP}^n is similarly defined as the set of lines through the origin in \mathbb{C}^{n+1} with coordinates x_0, \dots, x_n . Analogous to the description of \mathbb{CP}^1 as $\mathbb{C} \cup \{\infty\}$, here we have:

$$\mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1}$$

which we may write in coordinates as:

$$\mathbb{CP}^n = \{\lambda \cdot (1, y_1, \dots, y_n)\} \cup \{(0 : x_1 : \dots : x_n)\}$$

The former is our higher-dimensional analogue of $U_0 \subset \mathbb{CP}^1$. The full collection of sets $U_0, \dots, U_n \subset \mathbb{CP}^n$ and transition functions analogous to $y \mapsto y^{-1}$ may be worked out, but they require a better choice of coordinates which we will postpone until later.

Projective space is the environment in which projective varieties and, in particular, projective algebraic curves, are found as common zero loci of systems of homogeneous polynomial equations. We will study these in more detail later, but notice that a polynomial $F(x_0, \dots, x_n)$ that is a linear combination of monomials **of the same degree** d has the property that $F(\lambda \cdot (x_0, \dots, x_n)) = \lambda^d \cdot F(x_0, \dots, x_n)$, and thus while the **value** $F(x_0 : \dots : x_n)$ is not well-defined, the solutions to the equation $F(x_0 : \dots : x_n) = 0$ are well-defined as points of \mathbb{CP}^n .

Example 1.1. Consider the maps ϕ_d from \mathbb{CP}^1 to \mathbb{CP}^n given by:

$$\phi_d(s : t) = (s^d : s^{d-1}t : \dots : t^d)$$

Remark 1. All such maps are well-defined.

Remark 2. All such maps are injective (in fact, embeddings).

Remark 3. The images are cut out by quadratic equations.

The Conic. The image of ϕ_2 is the *plane conic*:

$$\{(s^2 : st : t^2) \mid (s : t) \in \mathbb{CP}^1\} \subset \mathbb{CP}^2$$

is precisely the set of solutions to the equation $x_0x_2 - x_1^2 = 0$.

The Twisted Cubic. The image of ϕ_3 is the *twisted cubic*:

$$C_3 = \{(s^3 : s^2t : st^2 : t^3) \mid (s : t) \in \mathbb{CP}^1\} \subset \mathbb{CP}^3$$

and this requires three quadratic equations to carve it out:

$$0 = x_0x_3 - x_1x_2 = x_0x_2 - x_1^2 = x_1x_3 - x_2^2$$

Any two of these equations will cut out C_3 with an extra line. E.g.

$$0 = x_0x_3 - x_1x_2 = x_0x_2 - x_1^2$$

is the set of points on $C_3 \cup L$ where L is line defined by $x_0 = x_1 = 0$.

The Rational Normal Curve C_d is cut out by quadratic equations obtained by the vanishing of all the 2×2 minors of the matrix:

$$\begin{bmatrix} x_0 & x_1 & \cdots & x_{d-1} \\ x_1 & x_2 & \cdots & x_d \end{bmatrix}$$

More generally, given homogeneous polynomials F_0, \dots, F_n of the same degree in s and t , then as long as the F_i do not simultaneously vanish at any point $(s : t) \in \mathbb{CP}^1$, we may use them to construct a map:

$$\phi(s : t) = (F_0(s, t) : \cdots : F_n(s, t))$$

whose image is always carved out by homogeneous equations. However, these maps need not be injective!

Exercise 1.3. Find homogeneous cubic polynomials that cut out the images of each of the following maps:

$$\phi(s : t) = (s^3 : s^2t : t^3) \text{ and } \psi(z : w) = (s^3 : s^2t + st^2 : t^3)$$

We turn next to rational functions and divisors.

Definition 1.3. The field of rational functions $K(\mathbb{CP}^1)$ of \mathbb{CP}^1 is:

$$\mathbb{C}(y_1) = \mathbb{C}(y_0)$$

which are two points of view of the same field. There is useful third point of view of this field in terms of the x_0, x_1 coordinate variables on \mathbb{C}^2 . Namely, regard $y_0 = x_0/x_1$ (or $y_1 = x_1/x_0$) as a ratio of x variables and “homogenize” a rational function in y_0 by:

$$\phi(y_0) = \frac{f(y_0)}{g(y_0)} = \frac{f(x_0/x_1)}{g(x_0/x_1)} = \frac{f(x_0/x_1) \cdot x_1^d}{g(x_0/x_1) \cdot x_1^d} = \frac{F(x_0, x_1)}{G(x_0, x_1)}$$

where F, G are homogeneous in x_0 and x_1 of the same degree. E.g.

$$\frac{y_0}{y_0^2 + 1} = \frac{x_0/x_1}{(x_0/x_1)^2 + 1} = \frac{(x_0/x_1) \cdot x_1^2}{((x_0/x_1)^2 + 1) \cdot x_1^2} = \frac{x_0x_1}{x_0^2 + x_1^2}$$

Definition 1.4. (a) A **divisor** on a complex curve C is:

$$D = \sum_{i=1}^n d_i p_i$$

a formal sum of points $p_i \in C$ with integer coefficients d_i which is a sum over **all** points of C with only finitely many nonzero coefficients.

- (b) A divisor D is **effective** if $d_i \geq 0$ for all i .
- (c) The **degree** of a divisor D is the finite sum $\sum_i d_i \in \mathbb{Z}$.
- (d) The **support** of a divisor D is the finite set $\{p_i \in C \mid d_i \neq 0\}$.

Any non-zero rational function on \mathbb{CP}^1 may be put in lowest terms:

$$\phi(y_0) = \frac{f(y_0)}{g(y_0)} = c \cdot \frac{\prod_i (y_0 - r_i)^{d_i}}{\prod_j (y_0 - s_j)^{e_j}} = c \cdot \frac{\prod_i (x_0 - r_i x_1)^{d_i}}{\prod_j (x_0 - s_j x_1)^{e_j}} \cdot x_1^{e-d}$$

for some $c \in \mathbb{C}^*$ and $d = \sum d_i$ and $e = \sum e_j$.

Then the **divisor of zeroes** of ϕ on \mathbb{CP}^1 is:

$$\text{div}(\phi) = \sum_i d_i \cdot (r_i : 1) - \sum_j e_j \cdot (s_j : 1) + (e - d) \cdot (1 : 0)$$

which is a divisor of degree zero that registers the order of zero or pole of ϕ at every point of \mathbb{CP}^1 , including the point at infinity!

Looking ahead. A projective non-singular curve C will come equipped with a field of rational functions $K(C)$ and a “div” map that produces a degree-zero divisor of zeroes and poles of any rational function ϕ (other than the zero function).

Definition 1.5. Divisors D and D' of the same degree on a curve C are *linearly equivalent* if there exists $\phi \in K(C)$ such that:

$$\text{div}(\phi) = D - D'$$

The *Mittag-Leffler* problem on an algebraic curve C is to determine whether, for a given effective divisor D' of degree d on C , there is a second effective divisor D such that $D \sim D'$. More generally, we want to know, given D' , the full set of effective divisors D satisfying $D \sim D'$. This is the *linear series* defined by D' , which is a *projective space*.

On a Riemann sphere the answer is straightforward.

Proposition 1.1. Every pair of effective divisors D and D' on \mathbb{CP}^1 of the same degree are linearly equivalent.

Proof. Write:

$$D = \sum_i d_i (r_i : 1) + a \cdot (1 : 0) \text{ and } D' = \sum_j e_j \cdot (s_j : 1) + b \cdot (1 : 0)$$

Then the rational function:

$$\phi = \frac{\prod_i (y_0 - r_i)^{d_i}}{\prod_j (y_0 - s_j)^{e_j}} = \frac{\prod_i (x_0 - x_1 r_i)^{d_i}}{\prod_j (x_0 - x_1 s_j)^{e_j}} \cdot x_0^{e-d}$$

has the desired property. \square

Remark. If $D \sim D'$ are a pair of linearly equivalent effective divisors on C with disjoint supports, then $\phi \in K(C)$ solving $\text{div}(\phi) = D - D'$ may be viewed as a *meromorphic* function on C mapping $\phi : C \rightarrow \mathbb{CP}^1$ to the Riemann sphere such that $\phi^{-1}(0 : 1) = D$ and $\phi^{-1}(1 : 0) = D'$ (after taking multiplicities into account)

More generally, $\phi^{-1}(s : t) = D_{(s:t)}$ are divisors (of degree d) indexed by points of \mathbb{CP}^1 that “interpolate” between D and D' .

Exhibiting C in this way as a branched cover of \mathbb{CP}^1 is analogous to choosing generators and relations for a group. It determines the curve, though (as in the case of a group) it is a highly non-trivial exercise in general to determine whether two curves “presented” in this way as branched covers of \mathbb{CP}^1 are isomorphic.

Notice, however, that if:

$$p \sim q \text{ on a curve } C$$

then the map

$$\phi : C \rightarrow \mathbb{CP}^1$$

is a bijection and, in fact, an isomorphism of algebraic curves. The Riemann-Roch theorem will tell us that every genus zero curve has a pair of linearly equivalent effective divisors of degree one, which implies that every genus zero curve is isomorphic to the Riemann sphere \mathbb{CP}^1 . The situation is completely different in higher genus, in which curves have *moduli* (continuous families of non-isomorphic curves).