Algebraic Geometry (Math 6130)

Utah/Fall 2016.

2. Projective Varieties. Classically, projective space was obtained by adding "points at infinity" to k^n. Here we start with projective space and remove a hyperplane, leaving us with an affine space. Even though they are not functions on projective space, there is a well-defined notion of vanishing of a homogeneous polynomial at a point of projective space. This allows us to define algebraic sets and varieties in projective space analogous to the algebraic sets and varieties in §1.

Let V be a finite-dimensional vector space over a field k and denote the dual space $\mathrm{Hom}(V,k)$ by V^*. In the 1960s and 70s, there was some trans-Atlantic controversy over whether the projective space associated to V ought be the set of minimal *subspaces* (lines through the origin) in V or else the set of minimal *quotient spaces* of V. We'll use both definitions here (with different fonts to distinguish them).

Definition 2.1. (i) (American projective space) Let $\mathrm{P}(V)$ be the set of lines through the origin in V, i.e.

$$\mathrm{P}(V) = (V - \{0\})/\sim$$

where $v \sim v'$ if if $v = \lambda v'$ for some $\lambda \in k^*$.

(ii) (European projective space) Let $\mathbb{P}(V)$ be the set of one-dimensional quotients of V, i.e.

$$\mathbb{P}(V) = (V^* - \{0\})/\sim$$

where $(V \stackrel{q}{\to} k) \sim (V \stackrel{q'}{\to} k)$ if there is a linear isomorphism $k \stackrel{\lambda}{\to} k$ (multiplication by $\lambda \in k^*$) such that $q = \lambda \circ q'$.

Remarks. (a) To each non-zero (hence surjective) map $V \stackrel{q}{\to} k$, we may associate the kernel $W = \ker(q) \subset V$, which is a maximal subspace, i.e. a hyperplane in V. Conversely, to each hyperplane $W \subset V$, the quotient $V \to V/W \cong k$ is uniquely determined up to the equivalence. So $\mathbb{P}(V)$ may be thought of as the set of hyperplanes in V.

(b) Clearly there is a natural identification $\mathrm{P}(V) = \mathbb{P}(V^*)$, but just as a vector space is not canonically isomorphic to its dual, there is no canonical isomorphism between $\mathrm{P}(V)$ and $\mathbb{P}(V)$. On the other hand, once an isomorphism $V \cong V^*$ is chosen (e.g. from a choice of basis) then an isomorphism between $\mathrm{P}(V)$ and $\mathbb{P}(V)$ results.

This bit of pedantry will be important when we replace vector spaces with vector bundles (and the reader may have noticed that we have not yet even defined an isomorphism of projective spaces).

Example 2.1. (a) $\mathrm{P}(k) = \mathbb{P}(k)$ is a point.

(b) A short exact sequence of vector spaces:

$$(*)\ 0 \to U \xrightarrow{f} V \xrightarrow{g} W (= V/U) \to 0$$

determines inclusions: $\mathrm{P}(U) \subset \mathrm{P}(V)$ and $\mathbb{P}(W) \subset \mathbb{P}(V)$.

These are the **projective subspaces** of $P(V)$ and $\mathbb{P}(V)$ respectively. Maximal projective subspaces are called **projective hyperplanes.** In particular, if $P(U) \subset P(V)$ is a hyperplane then W is a line, and in that case we lose no generality by letting $W = k$.

(c) Suppose $W = k$ in $(*)$. A map $l : k \to V$ *splits* the sequence $(*)$ if the composition $k \xrightarrow{l} V \xrightarrow{g} k$ is nonzero. A pair of splittings l, l' differ (after a uniquely determined scaling of l') by an element $l' - l : k \to U$. This gives the set of splittings the structure of an affine space with U as the underlying vector space.

In a bit more detail, a splitting l gives rise to an isomorphism:

$$(f, l) : U \oplus k \xrightarrow{\sim} V$$

The points of $P(V) - P(U)$ may be identified with maps $l' : k \to U \oplus k$ (using the isomorphism above) with $l'(1) = (u, 1)$. In other words, once the splitting l is chosen, we have: $P(V) - P(U) = U$ with the chosen splitting l representing the origin in U.

Without the choice of splitting, $P(V) - P(U)$ is "only" affine space.

This is important enough to record it as:

Proposition 2.1: Each point of $\mathbb{P}(V)$ yields a sequence:

$$(*)\ 0 \to U \to V \to k \to 0$$

and a projective hyperplane $P(U) \subset P(V)$. The complement of $P(U)$ is an affine space with underlying vector space U, which acquires an origin with a choice of $p \in P(V) - P(U)$ (splitting the sequence $(*)$).

Conversely, given a vector space U, we may let $V = U \oplus k$ and then:

$$P(U) \cup U = P(V)$$

is a disjoint union of U and a projective space of lines in U.

Definition 2.2. Projective n-space is $\mathbb{P}^n_k = P(k^{n+1})$. It contains:

coordinate hyperplanes $H_i = P(U_i)$ and their complements $U_i \subset \mathbb{P}^n_k$

for each $i = 0, ..., n$ corresponding to the split sequences:

$$0 \to U_i = \langle e_j \mid j \neq i \rangle \to k^{n+1} \to \langle e_i \rangle \to 0$$

Remarks. (a) Projective n-space is covered by subsets that we may identify with the vector spaces U_i:

$$\mathbb{P}_k^n = \bigcup_{i=0}^n U_i$$

with $[l : k \to k^{n+1}] \in U_i$ if and only if $l(1) = \sum_{j=0}^n a_j e_j$ with $a_i \neq 0$.

(b) Projective n-space is a disjoint union of vector spaces:

$$\mathbb{P}_k^n = U_0 \cup \mathbb{P}_k^{n-1} = U_0 \cup (U_{01} \cup \mathbb{P}_k^{n-2}) = \cdots = U_0 \cup U_{01} \cup \cdots \cup U_{0\ldots n}$$

where $U_{01\ldots,i} = \langle e_{i+1}, ..., e_n \rangle \subset k^{n+1}$ and in particular $U_{0\ldots n} = \{0\}$.

Warning! Do not confuse the subspaces $U_{0\ldots i} = U_0 \cap \cdots \cap U_i \subset k^{n+1}$ with the subsets $U_0 \cap \cdots \cap U_i \subset \mathbb{P}_k^n$. The former are vector spaces of dimension $n - i$, and the latter (as we will see) are all open, dense subsets of $\mathbb{P}_k^n$ of the ssame dimension n

Notation 2.1. If $l : k \to k^{n+1}$ with $l(1) = \sum_{i=0}^n a_i e_i \neq 0$, then:

$$(a_0 : \cdots : a_n) \in \mathbb{P}_k^n$$

are the **projective coordinates** of the point associated to l, with the colons between coordinates indicating the ambiguity:

$$(a_0 : \cdots : a_n) = (\lambda a_0 : \cdots : \lambda a_n) \text{ for } \lambda \in k^*$$

Next we revisit homogeneous polynomials in this context.

Observation. If $F \in S_d = k[x_0, ..., x_n]_d$ and $0 \neq \underline{a} \in k^{n+1}$, then

$$F(\underline{a}) \neq 0 \Leftrightarrow F(\lambda \underline{a}) = \lambda^d F(\underline{a}) = 0$$

for all $\lambda \in k^*$, hence $V(F) \subset k^{n+1}$ is a union of lines through the origin and more generally, $V(I) \subset k^{n+1}$ is a union of lines through the origin if $I = \langle F_1, \cdots F_m \rangle \subset S$ is a homogeneous (graded) ideal. Conversely,

Definition 2.3. A subset $C \subset V$ of a vector space is a **cone over the origin** if C is a union of lines through the origin (the *rulings* of C).

Proposition 2.2. The ideal $I(C) \subset S$ of a cone over the origin in k^{n+1} is homogeneous if k is an infinite field.

Proof. For $f \in S$, let $f = f_0 + \cdots + f_d$ be the decomposition of f into homogeneous terms. Then:

$$f(\lambda \underline{a}) = f_0(\underline{a}) + \lambda f_1(\underline{a}) + \cdots + \lambda^d f_d(\underline{a})$$

Thus if $f(\underline{a}) = 0$ for $\underline{a} \neq 0$, then $f(\lambda \underline{a})$ vanishes for all (infinitely many) values of $\lambda \in k$, i.e. it has infinitely many roots, as a polynomial in λ, so. it is the zero polynomial. It follows that if $f \in I(C)$, then each $f_e \in I(C)$, i.e. $I(C)$ is a homogeneous ideal.

Definition 2.4. Given a homogeneous ideal $I \subset S$, then:

$$V_h(I) = \{\text{rulings of the cone } V(I) \subset k^{n+1}\} \subset \mathbb{P}^n_k$$

is the associated **algebraic** set, and given $X \subset \mathbb{P}^n_k$, let $C(X) \subset k^{n+1}$ be the cone over the origin with rulings indexed by X, and:

$$I(X) := I(C(X)) \subset S$$

is the associated **geometric** homogeneous ideal (assuming k is infinite).

Convention. If $X = \emptyset$, let $C(X) = \{0\}$ and $I(X) = m = \langle x_0, ..., x_n\rangle$. This is often called the *irrelevant* homogeneous ideal. It is the only homogeneous ideal that is also a maximal ideal in the traditional sense.

Let k be algebraically closed (hence in particular, infinite).

Nullstellensatz for $\mathbb{P}^n_k$. Homogeneous ideals $I \subset \langle x_0, ..., x_n\rangle$ satisfy:

$$I(V_h(I)) = \sqrt{I}$$

Proof. This follows from the Nullstellensatz (or rather Corollary 1.2) whenever $V_h(I) \neq \emptyset$, and from the convention when $V_h(I) = \emptyset$, since we require that $I \subset \langle x_0, ..., x_n\rangle$ so that in that case, $V(I) = \{0\}$.

Corollary 2.1. $V_h(I) = \emptyset$ if and only if there is an $N > 0$ such that $x_i^N \in I$ for all $i = 0, .., n$.

Via the Nullstellensatz, we have "graded" correspondences:

$$X \leftrightarrow I(X) \leftrightarrow k[X]_\bullet := S/I(X)$$

among subsets of $\mathbb{P}^n_k$, homegeneous ideals in S and quotient rings of S:

algebraic sets $\leftrightarrow$ radical homogeneous ideals $\leftrightarrow$ reduced graded quotients

projective varieties $\leftrightarrow$ homogeneous primes $\leftrightarrow$ graded quotient domains

points $\leftrightarrow$ submaximal homogeneous ideals of the form $I = \langle l_1, .., l_n\rangle$

where $l_1, .., l_n \in S_1$ are n linearly independent forms.

The graded quotient ring $S \to R$ corresponding to a point has Hilbert function $\dim(R_d) = 1$ for all $d \geq 0$ and graded Koszul resolution:

$$0 \to S(-n) \to \cdots \bigoplus^{\binom{n}{2}} S(-2) \to \bigoplus^{n} S(-1) \to S \to R \to 0$$

Definition 2.5. (a) The **graded homogeneous coordinate ring** of a projective variety $X \subset \mathbb{P}^n_k$ is the graded domain:

$$k[X]_\bullet := k[x_0, ..., x_n]/I(X)$$

which coincides with the coordinate ring of the cone $C(X) \subset k^{n+1}$.

(b) The **field of rational functions** $k(X)$ of $X \subset \mathbb{P}^n_k$ is:

$$k(X) = \left\{ \frac{F}{G} \mid F, G \in k[X]_d \text{ for some } d,\ G \neq 0 \right\} \subset k(C(X))$$

Proposition 2.2. The fields $k(X)$ and $k(C(X))$ of rational functions associated to a (nonempty!) projective variety $X = V_h(P) \subset \mathbb{P}^n_k$ and the cone $C(X) = V(P) \subset k^{n+1}$ over the origin are related by:

$$k(C(X)) \cong k(X)(x)$$

Proof. Define the integer-graded ring:

$$R_\bullet = \left\{ \frac{F}{G} \mid F \in k[X]_d,\ G \in k[X]_e,\ G \neq 0 \right\} \subset k(C(X))$$

and choose $0 \neq x \in k[X]_1$ which is always possible if $X \neq \emptyset$. Then:

(i) $R_\bullet$ is isomorphic to $k(X)[x, x^{-1}]$ (graded by the power of x)

(ii) The ordinary field of fractions of $R_\bullet$ is $k(C(X))$.

from which the Proposition immediately follows.

Definition 2.6. Let $X \subset \mathbb{P}^n_k$ be a projective variety. Then:

(a) $\dim(X)$ is the transcendence degree of $k(X)$ over k.

(b) X is **nonsingular at** $a \in X$ if:

$$\text{rank}\left(\left(\frac{\partial F_i}{\partial x_j} \right)(a) \right) = n - \dim(X)$$

i.e. the rank of the Jacobian matrix agrees with the codimension of X in $\mathbb{P}^n_k$, where $P = I(X) = \langle F_1, ..., F_m \rangle$ (with homogeneous generators). The variety X is **nonsingular** if it is nonsingular at every point.

Remarks. (i) The field $k(a) = k$ when $a \in \mathbb{P}^n_k$ is a point!

(ii) By Proposition 2.2, the dimension of X satisfies:

$$\dim(X) + 1 = \dim(C(X))$$

where $C(X)$ is the associated cone in k^{n+1}.

(iii) The point $a \in X$ is nonsingular if and only if every nonzero point $\underline{a} \in C(X)$ in the ruling corresponding to a is nonsingular since the codimension of X in $\mathbb{P}^n_k$ agrees with the codimension of $C(X)$ in k^{n+1} and the Jacobian matrices are the same!

(iv) In the next section, we will cover projective varieties X with affine varieties $X \cap U_i$, and see that there, too, the definitions above are sensible if we think of the $X \cap U_i$ as being "local" affine charts of the "global" projective variety.

Definition 2.7. A nonsingular projective variety X is a **geometric model** for its function field $k(X)$.

Theorem (Resolution of Singularities): If k is algebraically closed of characteristic zero, then every finitely generated field K of finite transcendence degree over k has a nonsingular geometric model.

Remark. This is unknown when $\text{char}(k) = p$ and $\text{tr deg}(K/k) \geq 4$.

Birational algebraic geometry extracts information about a field K from its nonsingular geometric models.

We end with some basic examples from (multi)linear algebra.

Example 2.1 (Continued) A projective subspace $P(U) \subset \mathbb{P}^n_k$ coming from a sequence of vector spaces:

$$0 \to U \to k^{n+1} \to W \to 0$$

is a projective variety, cut out by the linear forms in $W^* \subset (k^{n+1})^*$. If we let $y_0, \ldots, y_m$ be a basis for $U^* = (k^{n+1})^*/W^*$, then:

$$k(P(U)) \cong k(y_1, ..., y_m)$$

and the cone over the origin in k^{n+1} with rulings indexed by $P(U)$ is, simply, $U \subset k^{n+1}$ itself, with $k(U) \cong k(y_0, ..., x_m)$.

Example 2.2. Quadrics. Consider a quadratic polynomial:

$$F = \sum_{i>j} c_{ij} x_i x_j \in S_2 = k[x_0, ..., x_n]_2$$

We may convert F into a symmetric matrix:

$$\Gamma_F := (\gamma_{ij}) \text{ with } \gamma_{ii} = c_{ii} \text{ and } \gamma_{ij} = \frac{1}{2} c_{ij} \text{ for } i \neq j$$

With this conversion, we see that:

$$F(a_0 : ... : a_n) = 0 \iff (\underline{a})^T \Gamma_F (\underline{a}) = 0$$

i.e. the points of the **quadric hypersurface** $Q = V(F) \subset \mathbb{P}^n_k$ are the lines $\lambda \underline{a} \subset k^{n+1}$ that are *isotropic* with respect to the inner product defined by Γ_F. In particular, if $U = \ker(\Gamma_F) \subset k^{n+1}$, then $P(U) \subset Q$. In fact, $P(U) \subset Q$ is the locus of singular points of the quadric Q. In particular, when Γ_F is invertible, tthe quadric Q is nonsingular.

Since we assume that k is algebraically closed, the Gramm-Schmidt process for the inner product yields a basis for k^{n+1} with respect to which the inner product diagonalizes with 1's and 0's on the diagonal. That is, with respect to this new basis $v_0, ..., v_n \in k^{n+1}$ and dual basis $w_0, ..., w_n \in (k^{n+1})^*$, $F = w_0^2 + \cdots + w_m^2$ and the singular locus of Q is the projective subspace $P(\langle e_{m+1}, ..., e_n \rangle)$.

We may also consider the projective space of quadrics. This is:

$$\mathbb{P}_k^{\binom{n+2}{2}-1}$$

for $k^{\binom{n+2}{2}} = \mathrm{Sym}^2(k^{n+1})$ with basis $e_i \otimes e_i$ and $e_i \otimes e_j + e_j \otimes e_i$ with dual basis $(x_{ij}; i < j)$, so this projective space is associated to the ring:

$$S = k[x_{ij}; i < j]$$

Via the Nullstellensatz, specifying a quadric $Q \subset \mathbb{P}_k^n$ is the same as specifying its homogeneous ideal $\langle \sum c_{ij}x_ix_j \rangle$ up to scalar multiple, so the quadrics are in bijection with he points $\mathbb{P}_k^{\binom{n+2}{2}-1}$ via:

$$Q \leftrightarrow (\gamma_{ij}) \in \mathbb{P}_k^{\binom{n+2}{2}-1}$$

with the exception of the polynomials $\sum c_{ij}x_ix_j$ that are perfect squares. But these are the image of the **Veronese embeddings**:

$$v_{n,2} : \mathbb{P}_k^n \to \mathbb{P}_k^{\binom{n+2}{2}-1}; \sum a_ie_i \mapsto (\sum a_ie_i)^{\otimes 2}$$

This image is a projective variety that is an intersection of quadrics.

Subexamples. (i) In the case $n = 1$, we have:

$$v_{1,2} : \mathbb{P}_k^1 \to \mathbb{P}_k^2; (a_0 : a_1) \mapsto (a_0^2 : a_0a_1 : a_1^2)$$

and the image is the conic $C = V(x_{00}x_{11} - x_{01}^2) \subset \mathbb{P}_k^2$.

(ii) In the case $n = 2$, we have the **Veronese surface**:

$$v_{2,2} : \mathbb{P}_k^2 \to \mathbb{P}_k^5; (a_0 : a_1 : a_2) \mapsto (a_0^2 : a_0a_1 : a_1^2 : a_0a_2 : a_1a_2 : a_2^2)$$

and the equations for the image are the quadrics appearing as the 2×2 minors of the following symmetric matrix:

$$A = \begin{bmatrix} x_{00} & x_{01} & x_{02} \\ x_{01} & x_{11} & x_{12} \\ x_{02} & x_{12} & x_{22} \end{bmatrix}$$

Here are a couple of interesting features of this example:

(i) There are five independent quadratic polynomials among the 2×2 minors of A, and these generate the homogeneous ideal $I(v_{2,2}(\mathbb{P}_k^2))$, yet $v_{2,2}(\mathbb{P}_k^2) \subset \mathbb{P}_k^5$ has codimension three. In general, a homogeneous ideal $P \subset S$ requires more generators than the codimension of the variety $X = V(P) \subset \mathbb{P}_k^n$. The rare homogeneous ideals P that are generated by $\mathrm{codim}(X)$ generators are called **complete intersection ideals**.

(ii) The cubic hypersurface in $\mathbb{P}_k^5$ cut out by $G = \det(A)$ parametrizes the locus of singular quadrics in $\mathbb{P}_k^2$, together with the image of $v_{2,2}$. This is itself a singular hypersurface, singular along the image of $v_{2,2}$!

Example 2.3 d-uple embeddings generalize the Veronese embeddings. Let $V = k^{n+1}$, and consider the mapping:

$$v_{n,d} : P(V) \to P(\mathrm{Sym}^d V);\ v_{n,d}(\sum a_i e_i) = (\sum a_i e_i)^{\otimes d}$$

to the space of symmetric tensors. With the natural basis for:

$$\mathrm{Sym}^d(V) \subset V^{\otimes d}$$

we have:

$$v_{n,d}(\cdots : a_i : \cdots) = (\cdots : \prod_{i=0}^{n} a_i^{d_i} : \cdots) \text{ for } d_0 + \cdots + d_n = d$$

i.e. it is given by the collection of a monomials of degree d in the a_i. From this, it is not difficult to see that the image of $v_{n,d}$ is always cut out by quadratic polynomials, and indeed the homogeneous ideal $I(v_{n,d}(\mathbb{P}_k^n))$ is generated by quadratic polynomials.

The case $n = 1$ is particularly nice. Here:

$$v_{1,d}(a_0 : a_1) = (a_0^d : a_0^{d-1} a_1 : \cdots : a_0 a_1^{d-1} : a_1^d)$$

and the image is the rank one locus of the matrix:

$$\begin{bmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

The image of $v_{1,d}$ in $\mathbb{P}_k^d$ is the **rational normal curve** of degree d.

Example 2.4 Segre embeddings are of the form:

$$s_{V,W} : P(V) \times P(W) \to P(V \otimes W);\ (\lambda \underline{a}, \mu \underline{b}) \mapsto \lambda\mu \underline{a} \otimes \underline{b}$$

In coordinates, if $V = k^{m+1}$ and $W = k^{n+1}$, then:

$$s_{m,n}((a_0 : ... : a_m), (b_0 : \cdots b_n)) = (\cdots a_i b_j \cdots)$$

and once again the image of $s_{m,n}$ is cut out by quadratic polynomials and the homogeneous ideal is generated by quadratic polynomials.

For example, the image of $s_{1,1} : \mathbb{P}_k^1 \times \mathbb{P}_k^1 \to \mathbb{P}_k^3$ is:

$$\left\{(a_0 b_0 : a_1 b_0 : a_0 b_1 : a_1 b_1) \in \mathbb{P}_k^3\right\} = V(x_0 x_3 - x_1 x_2)$$

which is "the" nonsingular quadric in $\mathbb{P}_k^3$.

Segre embeddings can be generalized to finite sets of vector spaces:

$$s_{V_1,\ldots,V_m} : P(V_1) \times \cdots \times P(V_m) \to P(V_1 \otimes \cdots P(V_m)$$

in the obvious way.

Example 2.5. The Grassmannian Let V be a vector space over k of dimension n. Then $\mathbb{G}(m, V)$ is the subset:

$$\mathbb{G}(m, n) \subset P(\wedge^m V)$$

of decomposable elements. That is, points of $\mathbb{G}(m, n)$ are of the form:

$$(*) \quad \sum_{\sigma \in \Sigma_m} (-1)^{\mathrm{sgn}(\sigma)} v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(m)};\ v_1, ..., v_m \in V$$

for linearly independent vectors $v_1, ..., v_m \in V$.

The points of the Grassmannian are in a natural bijection with the set of subspaces of V, so that for example:

$$\mathbb{G}(m, V) = P(V) \text{ and } \mathbb{G}(n-1, V) = \mathbb{P}(V)$$

Once again, the Grassmannian is cut out by (**Plücker**) quadrics both set-theoretically and ideal-theoretically, and the Grassmannians $\mathbb{G}(m, V)$ are projective varieties. This involves some combinatorics, but the case $m = 2$ and $\dim(V) = 4$ is particularly simple. Here,

$$\mathbb{G}(2, 4) \subset P(\wedge^2 V)$$

is a subset of the space of skew-symmetric maps, and:

$$\mathbb{G}(2, 4) \subset P\left(\begin{bmatrix} 0 & \gamma_{12} & \gamma_{13} & \gamma_{14} \\ -\gamma_{12} & 0 & \gamma_{23} & \gamma_{24} \\ -\gamma_{13} & -\gamma_{23} & 0 & \gamma_{34} \\ -\gamma_{14} & -\gamma_{24} & -\gamma_{34} & 0 \end{bmatrix}\right)$$

is the quadric cut out by the **Pfaffian** of the matrix of indeterminates:

$$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}$$

The Grassmannian $\mathbb{G}(m, V)$ is the image of the mapping

$$V^m \supset U \to P(\wedge^m V)$$

defined by $(*)$ above. This is well-defined on the subset: $U \subset V^m$ of linearly independent m-tuples of vectors, which is the complement of an algebraic subset of $V^m = (k^n)^m$.

Exercises.

2.1. Consider the inclusion: $k^n = \langle e_1, ..., e_n\rangle \subset \mathbb{P}^n_k$ associated to:

$$0 \to U_0 \to k^{n+1} \to k \to 0$$

and the splitting $e_0 : k \to k^{n+1}$. This geometry is associated to the process of *homogenizing* and *dehomogenizing* polynomials:

Deomogenizing. The dehomogenization of $F \in S_d$ is:

$$f(x_1, ..., x_n) := F(1, x_1, ..., x_0)$$

Homogenizing. For each $f \in k[x_1, ..., x_n]$ and $d \geq \deg(f)$,

$$F(x_0, ..., x_n) := x_0^d f\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right) \in S_d$$

produces a homogeneous polynomial that dehomogenizes to f.

(a) Show that dehomogenizing and homogenizing all the elements of an ideal converts homogeneous ideals in S to homogeneous ideals in $k[x_1, ..., x_n]$ and vice versa. Notice that on the level of ideals, there is no ambiguity....all values of $d \geq \deg(f)$ must be chosen when homogenizing an ideal containing f.

(b) Show that radical ideals and prime ideals are preserved under homogenizing and dehomogenizing. Moreover, show that:

$$I \subset k[x_1, ..., x_n] \mapsto \mathrm{hom}(I) \subset S \mapsto \mathrm{dehom}(\mathrm{hom}(I)) \subset k[x_1, ..., x_n]$$

returns the ideal I. What happens to a homogeneous ideal under:

$$I \subset S \mapsto \mathrm{dehom}(I) \subset k[x_1, ..., x_n] \mapsto \mathrm{hom}(\mathrm{dehom}(I) \subset S$$

2.2. (a) Under the bijections between ideals and varieties, we see from Exercise 2.1 (b) that the set of algebraic subsets of k^n is in bijection (via homogenization) with a subset of the set of algebraic subsets of $\mathbb{P}^n_k$. What geometric property must an algebraic subset of $\mathbb{P}^n_k$ have in order to be in the image of this identification?

(b) Consider the twisted cubic curve in k^3:

$$C = \{(s, s^2, s^3) \mid s \in k\} \subset k^3$$

Identify the curve $\overline{C} \subset \mathbb{P}^3_k$ associated to C via homogenizing, and show that $I(C)$ requires only two generators, whereas $I(\overline{C}) = \mathrm{hom}(I)$ requires three generators. Thus, it is **not sufficient** to homogenize the generators of an ideal to find the generators of the homogenized ideal.

2.3. (a) If $X \subset k^n$ is a variety and $\overline{X} \subset \mathbb{P}^n_k$ is the associated projective variety, find an isomorphism between the field of fractions:

$$k(X) \cong k(\overline{X})$$

(b) Show that a point $x \in X$ is non-singular as a point of X if and only if it is non-singular when viewed as a point $x \in \overline{X}$ of $\overline{X}$.

Hint: Try it first for a hypersurface. Euler's identity may be useful:

$$dF = \sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} \text{ for } F \in S_d$$

(c) Consider the nonsingular curve:

$$C = V(y^2 - \prod_{i=1}^{d}(x - \lambda_i)$$

for distinct roots $\lambda_1, ..., \lambda_d$. Find the complement $\overline{C} - C \in \mathbb{P}^2_k$ and decide whether the points of the complement are singular or not.

(d) Find $\overline{C}$ for $C = V(y - x^3)$ and show that it has a singular point.

2.4. Show that the singular locus of the quadric $Q = V(F) \in \mathbb{P}^n_k$ for $F \in S_2 - \{0\}$ is precisely the projective subspace $P(\ker(\Gamma_F))$, where Γ_F is the associated symmetric matrix.

2.5. Show that the images of the d-uple and Segre embeddings are non-singular projective varieties. Show that each $\mathbb{G}(m, V)$ is a variety. Show that the ideals of each of these is generated by quadratic polynomials.

2.6. Let G be the determinant of the matrix:

$$A = \begin{bmatrix} x_{00} & x_{01} & x_{02} \\ x_{01} & x_{11} & x_{12} \\ x_{02} & x_{12} & x_{22} \end{bmatrix}$$

Show that the singular locus of $X = V(G)$ is the Veronese surface.