Algebraic Geometry (Math 6130)

Utah/Fall 2016.

4. Some Properties of Maps. The equivalence of categories between affine varieties and k-algebra domains means that morphisms in the two categories contain the same information. Since every rational map of varieties is locally a regular map of affine varieties, the "algebra" of homomorphisms of k-algebras governs the local "geometry" of maps of varieties. In this section, we investigate some implications of this.

Injective Homomorphisms $\leftrightarrow$ Dominant Maps. Let

$$f : k[Y] \to k[X]$$

be an injective homomorphism of k-algebra domains. Then the image of the corresponding map of affine varieties:

$$\Phi = \mathrm{mspec}(f) : X \to Y$$

is (Zariski) dense, and conversely, a regular map of affine varieties with a dense image corresponds to an injective map of coordinate rings. Indeed, $h \in \ker(f)$ if and only if $U_h \cap \mathrm{im}(\Phi) = \emptyset$, and the equivalence follows since open sets of the form U_h are a basis for the topology.

Examples. (a) The inclusion $k[X] \subset k[X][h^{-1}]$ corresponds to the inclusion of the basic open set $U_h \subset X$, which is dense.

(b) The inclusion $k[x] \subset k[x, y]$ corresponds to the projection map $\pi_1 : k^2 \to k$, which is surjective.

(c) The homomorphism $f : k[x, y] \to k[s, t]$; $f(x) = s, f(y) = st$ is injective and corresponds to the map $\Phi : k^2 \to k^2$ with:

$\Phi(0, *) = (0, 0)$ (i.e. the y-axis collapses to the origin)

$\Phi^{-1}(a, b) = (a, b/a)$ whenever $a \neq 0$.

Thus $\Phi(k^2) = \{(0, 0)\} \cup U_x \subset k^2$ is dense, but neither open nor closed.

We may capture the geometry with the following definition:

Definition 4.1. A rational map $\Phi : X - - > Y$ of **varieties** is **dominant** if the image of the domain of Φ is dense in Y.

Observation. Every rational map of varieties is locally a regular map of affine varieties. That is, for each $y \in \mathrm{im}(\Phi)$ and $x \in \Phi^{-1}(y)$, we can find affine neighborhoods $y \in U$ and $x \in V$ such that:

$$\Phi|_V : V \to U$$

is a regular map of affine varieties. This simply follows from the fact that the affine open subsets form a basis for the topology of a variety.

Proposition 4.1. A rational map $\Phi : X - - > Y$ is dominant if and only if each local map

$$\Phi|_V : V \to U$$

of affine varieties corresponds to an injective map $\Phi^* : k[U] \to k[V]$.

Proof: Since varieties are irreducible Noetherian spaces and the domain of Φ is open and nonempty, it follows that if the image of Φ is dense, then the image of $\Phi|_V$ is dense for any $V \subset X$ and we have reduced to the affine case.

Dominant rational maps "are" inclusions of rational function fields:

Definition 4.2. The field of rational functions $k(X)$ of a variety X is:

$$k(X) = \lim_{V \subset X} \mathcal{O}_X(V) \text{ (the inverse limit)}$$

i.e. it is the ring of functions that are regular **somewhere** on X.

It is an exercise to see that for each open affine $V \subset X$,

$$k(V) = k(X), \text{ where } k(V) \text{ is the field of fractions of } k[V]$$

and hence in particular that $k(V)$ is independent of V.

Corollary 4.1. The data of a **dominant** rational map from X to Y is equivalent to an inclusion of their fields of rational functions:

$$k(Y) \subset k(X)$$

Proof. The inclusion of fields associated to a dominant rational map is obtained locally, since the maps on coordinate functions are injective. Conversely, an inclusion of fields induces a dominant rational map on affine varieties, via the map $k[U] \subset k(U) \subset k(V)$, and this extends, by definition, to a rational map of varieties.

Surjective Homomorphisms $\leftrightarrow$ Closed Embeddings. A surjective k-algebra homomorphism:

$$f : k[Y] \to k[X]$$

factors through:

$$k[Y] \to k[Z] \cong k[X]$$

where $Z \subset Y$ is the closed irreducible subset defined by $P = \ker(f)$. Thus X is isomorphic to $Z \subset Y$.

Example. Each surjection $k[y_1, ..., y_n] \to k[Y]$ corresponds to $Y \subset k^n$.

To extend this geometry, we need to carefully explain what we mean when we say that a closed irreducible subset $Z \subset Y$ of a variety is itself a variety. In case Y is affine, then $Z \subset Y$ corresponds to $k[Z] = k[Y]/I(Z)$, and it is clear what is meant.

Proposition 4.2. Let Y be a variety and let $Z \subset Y$ be a closed irreducible subset. Then Z has the structure of a variety, with:

- the induced (Noetherian) topology
- the field of rational functions on Z defined by:

$$k(Z) := \mathcal{O}_{Y,Z}/m_Z$$

where $\mathcal{O}_{Y,Z} \subset k(X)$ is the subring consisting of rational functions that are defined **somewhere** on Z, and m_Z is the maximal ideal of such rational functions that are identically zero on their domain in Z.

- the sheaf $\mathcal{O}_Z(U)$ is defined by:

$$\mathcal{O}_Z(U) = \{\phi \in k(Z) \mid \phi(y) \text{is defined for all } y \in U\}$$

Proof. When restricted to an open affine $W \subset Y$, this structure on $Z \cap W$ coincides with the structure of $Z \cap W \subset W$ as a closed affine subvariety of W. Thus Z is locally affine, and it is an exercise to see that Z is separated, hence a variety.

Definition 4.3. A regular map $\Phi : X \to Y$ of varieties is a **closed embedding** if it factors through an isomorphism $\Phi : X \tilde{\to} Z$ for $Z \subset Y$ an irreducible closed subset with the induced variety structure.

It is also an exercise to see that a closed embedding $\Phi : X \to Y$ of varieties is locally a closed embedding, $\Phi|_V : V \to U$, of affine varieties, corresponding to surjective maps $(\Phi|_V)^* : k[U] \to k[V]$.

For a third relation between algebraic and geometric maps, we use:

Definition 4.4. A homomorphism $f : B \to A$ of commutative rings is **integral** if A is finitely generated as a B-module (via f).

Remark. Since f is clearly integral if and only if $\overline{f} : B/\ker(f) \to A$ is integral, we will usually reduce to injective homomorphisms.

Non-example. The inclusion $k[X] \to k[X][h^{-1}]$ corresponding to the open inclusion $U_h \subset X$ of affine varieties is **not** integral, since $k[X][h^{-1}]$ cannot be finitely generated as a module over $k[X]$.

An Important Example. Let k be an infinite field (not necessarily algebraically closed) and let $A = k[x_1, ..., x_n]/P$ be an integral domain of transcendence degree m over k. Then there are:

$$y_i = \sum_{i=1}^{n} a_{i,j} x_i; \; j = 1, ..., m$$

such that the homomorphism $f : k[y_1, ..., y_m] \to k[X]$ is injective and integral. This is the **Noether Normalization Theorem.**

Proof. By induction on $n-m$. If $m=n$, then $P=0$ and there is nothing to prove. If $m<n$ then $\overline{x}_1, ..., \overline{x}_n \in A$ satisfy a relation:

$$f(\overline{x}_1, ..., \overline{x}_n) = 0$$

and if it were the case that $f = \overline{x}_n^d + \{\text{lower order in } \overline{x}_n\}$ we'd be done by induction since $1, \overline{x}_n, ..., \overline{x}_n^d$ would generate A as a module over

$$B = k[x_1, ..., x_{n-1}]/P \cap k[x_1, ..., x_{n-1}]$$

which is necessarily of the same transcendence degree over k as A. On the other hand, if f is not of this form, then we may replace:

$$y_i = x_i + a_i x_n \text{ for } a_i \in k \text{ and } i < n$$

and get $f(\overline{y}_1, ..., \overline{y}_{n-1}, \overline{x}_n) = g(a_1, ..., a_{n-1})\overline{x}_n^d +$ {lower order in $\overline{x}_n$} for some non-zero polynomial $g(a_1, ..., a_{n-1})$. Since k is infinite, we may choose $a_1, ..., a_{n-1} \in k$ so that $g(a_1, ..., a_{n-1}) \neq 0$ and then proceed with our induction with the new variables $y_1, ..., y_{n-1}$.

Integral Homomorphisms $\leftrightarrow$ Finite Maps.

Definition 4.5. A morphism of affine varieties: $\Phi : X \to Y$ is **finite** if the associated homomorphism Φ^* is integral.

Example. The Noether Normalization example above is both finite and dominant. It is also the restriction of the *linear projection*

$$\pi : k^n \to k^m; \quad \text{defined by } \pi(b_1, ..., b_n) = (...., \sum a_{i,j} b_j, ...)$$

so the theorem implies that each embedded affine variety $X \subset k^n$ of dimension m projects via a finite and dominant map to k^m.

Theorem 4.1. If $\Phi : X \to Y$ is a finite map of affine varieties, then:

(a) $\Phi^{-1}(y)$ is a finite set, for all $y \in Y$.

(b) Φ maps closed sets to closed sets. In particular, if Φ is also dominant, then Φ is surjective.

Proof. The closure of the image $Z = \overline{\Phi(X)} \subset Y$ corresponds to the prime ideal $\ker(\Phi^*) \subset k[Y]$, and we lose no generality by replacing Y with Z and assuming that Φ is also dominant.

Let $f = \Phi^* : k[Y] \to k[X]$ be the associated integral map. Then via the identification $Y \leftrightarrow \text{mspec}(k[Y])$, we have:

$$\Phi^{-1}(m_y) = \{m_x \subset k[X] \mid m_x \supset f(m_y)\}$$

so the set $\Phi^{-1}(m_y)$ is in bijection with the set of maximal ideals in:

$$A = k[X]/\langle f(m_y) \rangle$$

where $\langle f(m_y) \rangle$ is the ideal generated by the set $f(m_y) \subset k[X]$.

Since f is integral, it follows that $\overline{f} : k[Y]/m_y \to A$ is also integral, hence A is finitely generated as a vector space over $k[Y]/m_y = k$. But an algebra over k that has dimension n as a vector space over k has at most n maximal ideals (Exercise). This proves (a).

Next, it is enough to show that Φ maps *irreducible* closed subsets of X to (necessarily irreducible) closed subsets of Y, and indeed given such a subset $W = V(P) \subset X$, we may restrict Φ to W and obtain a map $\Phi|_W : W \to V = \overline{\Phi(W)}$, which is finite and dominant, corresponding to the integral ring homomorphism $\overline{f} : k[Y]/f^{-1}(P) \to k[X]/P$. In other words, it suffices to show that every finite and dominant map of affine varieties is surjective.

Looking back at the proof of (a), we simply need to show that $A \neq 0$. This is accomplished by "going up." Consider $m_y \subset k[Y]$ again and:

$$S = k[Y] - m_y \text{ as a multiplicative system in both } k[Y] \text{ and } k[X]$$

(the latter via the inclusion $k[Y] \subset k[X]$). Then there is an inclusion:

$$k[Y]_S = k[Y]_{m_y} \subset k[X]_S$$

of the local ring $\mathcal{O}_{Y,y} = k[Y]_{m_y}$ in the more mysterious ring $k[X]_S$. But $k[X]_S$ is integral over $k[Y]_{m_y}$, and from this it follows that:

Lemma 4.1. Each maximal ideal $m \subset k[X]_S$ satisfies:

$$m \cap k[Y]_S = m_y \cdot k[Y]_{m_y}$$

Proof. It suffices to show that the ring $B = k[Y]_S/(m \cap k[Y]_S)$ is a field, since $k[Y]_S = k[Y]_{m_y}$ is a local ring with unique maximal ideal $m_y \cdot k[Y]_{m_y}$. So let $0 \neq \phi \in B$ and suppose ϕ is **not** invertible in B. Then because ϕ **is** invertible in $k[X]_S/m$, we have proper inclusions:

$$B \subset \phi^{-1}B \subset \phi^{-2}B \subset \cdots \subset k[X]_S/m$$

producing an infinite ascending chain of submodules of $k[X]_S/m$. But $k[X]_S/m$ is **finitely generated** as a module over B, contradicting the fact that any localization $k[Y]_S$ of a Noetherian ring is Noetherian. □

Returning to (b), we have found a maximal ideal $m \subset k[X]_S$ whose intersection with $k[Y]_S$ is m_y, and it follows from the correspondences of prime ideals under localization that the corresponding maximal ideal $m_x \subset k[X]$ satisfies $m_x \cap k[Y] = m_y$, as desired. □

Non-Example. Consider the non-integral inclusion of rings

$$k[x] \subset k[x, x^{-1}]$$

corresponding to the non-finite open embedding $k^* \subset k$.

Letting $S = k[x] - \langle x\rangle$, we obtain $k[x]_{\langle x\rangle} \subset k[x,x^{-1}]_S = k(x)$, and because $k(x)$ is not a finitely generated module over $k[x]_{\langle x\rangle}$, there is no contradiction in the infinite chain:

$$k[x]_{\langle x\rangle} \subset x^{-1}k[x]_{\langle x\rangle} \subset x^{-2}k[x]_{\langle x\rangle} \subset \cdots \subset k(x)$$

Indeed, $\phi = x$ is not invertible, and there is no ideal in the field $k(x)$ that intersects $k[x]_{\langle x\rangle}$ in the maximal ideal, reflecting the fact that $0 \in k$ is not in the image of the inclusion $k^* \subset k$.

Remark. The same strategy gives a proof of the **Nullstellensatz** from Noether Normalization. Namely, if $m \subset k[x_1, ..., x_n]$ is a maximal ideal, consider the field extension:

$$k \subset K = k[x_1, ..., x_n]/m$$

If K had transcendence degree $d > 0$ over k, then Noether Normalization would give an integral inclusion of rings: $k[y_1, ..., y_d] \subset K$ and then every polynomial $f \in k[y_1, .., y_d]$ would have an inverse in $k[y_1, ..., y_d]$ because otherwise:

$$k[y_1, ..., y_d] \subset f^{-1}k[y_1, ..., y_d] \subset f^{-2}k[y_1, ..., y_d] \subset \cdots \subset K$$

would be an infinite ascending chain of submodules of a finitely generated module. This is absurd, since polynomials of positive degree do not have inverse polynomials. We therefore conclude that K must be an algebraic extension of k and the Nullstellensatz follows.

Warning. Unlike injective and surjective ring homomorphisms, the geometric properties of a finite map in Theorem 4.1 are **not** equivalent to integrality of the corresponding homomorphism. This explains why the definition of a finite map of varieties isn't made in geometric terms.

Definition 4.6. (a) A map $\Phi : X \to Y$ of varieties is **affine** if the inverse image of every open affine subvariety $U \subset Y$ is an affine variety.

(b) An affine map Φ as above is **finite** if for each open affine $U \subset Y$ the map $\Phi|_V : V = \Phi^{-1}(U) \to U$ is a finite map of affine varieties.

In practice, these properties are checkable because of the following:

Theorem 4.2. To conclude that a map is affine or finite, it suffices to check the property for a single open affine cover $Y = \bigcup U_i$.

We use a **Criterion for Affineness** which is of independent interest. If V is a variety with $h_i, g_i \in \Gamma(V, \mathcal{O}_V)$; $i = 1, ..., n$ such that:

(i) $V_i = \{x \in V \mid h_i(x) \neq 0\}$ are affine open sets that cover V and

(ii) $\sum g_i h_i = 1$.

Then V is an affine variety.

Proof (of the criterion). Let $k[V_i] = k[x_{i,1}, ..., x_{i,m_i}]/P_i$, and let:

$$A = \Gamma(V, \mathcal{O}_V) = k[V_1] \cap \cdots \cap k[V_n] \subset k(V)$$

Each $V_i \cap V_j$ is a basic open affine subset of V_j (and V_i) since:

$$V_i \cap V_j = V_j - V(\rho_{V,V_j} h_i) = (V_j)_{h_i}$$

Therefore if $\phi \in k[V_i]$ then $\phi \in k[V_i \cap V_j] = k[V_j][h_i^{-1}]$ for each j, and $\phi h_i^{n_j} \in k[V_j]$ for some n_j. If we let $n = \max\{n_j\}$, then $\phi h_i^n \in A$. Thus:

$$(*)\ k[V_i] = A[h_i^{-1}] \text{ for each } i$$

In particular, each of the generators of $k[V_i]$ as a k-algebra satisfies:

$$x_{i,l} h_i^{n_{i,l}} \in A$$

and we may choose $n = \max\{n_{i,l}\}$ to make the power uniform over all generators. We now claim that $\{g_i, h_i, x_{il} h_i^n\}$ generates A as a k-algebra. To see this, write $a \in A$ as a polynomial in the generators of each of the rings $k[V_i]$:

$$a = p_i(x_{i_1}, ..., x_{i,m_i}) \in k[V_i]$$

and notice that the product ah_i^N for a sufficiently large N makes each $ah_i^N = p_i h_i^N$ expressible as a polynomial in the $x_{i,l} h_i^n$ and h_i. Now use:

$$a \cdot \left(\sum g_i h_i\right)^{(N-1)(n+1)} = a \cdot 1 = a$$

to conclude that a is expressible as a polynomial in the ah_i^N and g_i, h_i, hence also in $x_{i,l} h_i^n$ and g_i and h_i, as desired. Thus A is the image of:

$$k[y_{i,l}, z_i, w_i] \to k(V);\ y_{i,l} \mapsto x_{i,l} h_i^n, z_i \mapsto g_i, w_i \mapsto h_i$$

and now it is straightforward to conclude from $(*)$ that $V = \text{mspec}(A)$.

Proof of the Theorem. If $U, U' \subset Y$ are affine open subsets, then their intersection is covered by affines $U_h = U'_{h'}$ that are *simultaneously* basic open affine subsets of U and U'. To see this, first cover $U \cap U' = \bigcup U_{h_i}$ for $h_i \in k[U]$ and then cover each $U_{h_i} = \bigcup U'_{h'_{ij}}$ for $h'_{ij} \in k[U']$. But then $h'_{ij} \in k[U_{h_i}]$, so $h_i^n h'_{ij} \in k[U]$, and $U'_{h'_{ij}} = U_{h_i^{n+1} h'_{ij}}$, as desired.

Now, suppose $Y = \bigcup U_i$ is an open cover such that $V_i = \Phi^{-1}(U_i)$ is affine for all i. Let $U \subset Y$ be another open affine subset, and cover $U \cap U_i$ by simultaneous basic open affines $U_{h_{i,l}} = (U_i)_{h'_{i,l}}$. It follows from the fact that $(U_i)_{h'_{i,l}} \subset U_i$ is a basic open that its inverse image in Y is affine. It also follows from the fact that they cover U that there are regular functions $g_{i,l} \in k[U]$ such that $\sum g_{i,l} h_{i,l} = 1$. Now we may apply the criterion for affineness to $V = \Phi^{-1}(U)$ and the cover by the affine opens $\Phi^{-1}(U_{h_{i,l}})$. This takes care of the affine maps.

Next, suppose additionally that each of the maps $\Phi|_{V_i}$ is finite and let $a_{i,s} \in k[V_i]$ generate it as a module over $k[U_i]$. For each $h' \in k[U_i]$, the same elements will generate $k[(V_i)_{h'}]$ as a module over $k[(U_i)_{h'}]$. Given $U \subset Y$ and the cover by simultaneous basic open affines $U_{h_{i,l}}$, consider the elements $a_{i,s}h_{i,l}^n \in k[V]$ for a large enough value of n. These will belong to $k[V]$ **and** generate it as a module over $k[U]$ by virtue of:

$$a\left(\sum h_{i,l}g_{i,l}\right)^N = a$$

for large enough values of N (Exercise!).

Example (a) The projections $\pi : \mathbb{P}^n_k - V(x_0, ..., x_m) \to \mathbb{P}^m_k$ onto the first $m+1$ coordinates are affine since each $\pi^{-1}(U_i) = V_i$ is affine.

(b) If $Z \subset \mathbb{P}^n_k$ is closed and irreducible and $Z \cap V(x_0, ..., x_m) = \emptyset$, then $\pi|_Z : Z \to \mathbb{P}^m_k$ is a finite map. In particular, if $\dim(Z) = m$, then $\pi|_Z$ is finite and dominant. This is a projective version of the Noether Normalization Theorem.

We end this section by studying the fibers of a dominant map:

$$\Phi : X \to Y$$

More generally, if $W \subset Y$ is closed and irreducible (closed subvariety), then we are interested in the dimensions of the irreducible components $Z_1 \cup \cdots \cup Z_n = \Phi^{-1}(W)$ of the inverse image of W.

For example, consider again Example (c) from the top of this section. In this example, the map $\Phi : k^2 \to k^2$ has the following inverse images:

(a) The inverse image of a point p is either:
  (i) empty, if p is on the y-axis minus the origin.
  (ii) a point if p is off the y-axis.
  (iii) the y-axis if p is the origin.

(b) The inverse images of a line l in k^2 is:
  (i) an irreducible curve if l does not contain the origin
  (ii) the y-axis (mapping to the origin) if l is the y-axis
  (iii) the union of a horizontal line and teh y-axis if l contains the origin but is not the y-axis.

The following Proposition gets us started toward this goal.

Proposition 4.1. (a) If $W \subset Y$ is a closed subvariety of codimension c in an affine variety, then there are $f_1, ..., f_c \in I(W)$ such that all components of $V(f_1, ..., f_c)$ (including W) have codimension c in Y.

(b) On the other hand, if $g_1, ..., g_c \in \Gamma(X, \mathcal{O}_X)$ for a variety X, then each irreducible component of $V(g_1, ..., g_c)$ has codimension $\leq c$ in X.

Proof. We prove (a) by induction. If $f_1, ..., f_b \in I(W)$ are chosen so that every component of $V(f_1, ..., f_b)$ has codimension $b < c$, then for each such component Z_i, choose $p_i \in Z_i - W$. Then there is an $f_{b+1} \in I(W)$ that does not vanish at any of the p_i. By Krull's Theorem, every component of $V(f_1, ..., f_{b+1})$ then has codimension $b + 1$.

For (b), we also use induction. If Z is a component of $V(g_1, ..., g_c)$, then Z is contained in some component Z' of $V(g_1, ..., g_{c-1})$. We may assume that Z' has codimension $\leq c - 1$ in X, and then by Krull, $Z \subset Z' \cap V(g_c)$ has codimension 0 or 1 in Z', hence $\leq c$ in X.

Corollary 4.2. Let $\Phi : X \to Y$ be a dominant map of varieties, and

$$r = \dim(X) - \dim(Y)$$

If $W \subset Y$ is a closed subvariety, let Z be an irreducible component of $\Phi^{-1}(W)$ that dominates W. Then $\dim(Z) \geq \dim(W) + r$, i.e.

$$\text{codimension of } Z \text{ in } X \ \leq \ \text{codimension of } W \text{ in } Y$$

Note that as a special case, each irreducible component $Z \subset \Phi^{-1}(y)$ of each *fiber* of Φ has dimension at least $r = \dim(X) - \dim(Y)$.

Proof. Replacing Y with an open affine $U \subset Y$ that intersects W, we lose no generality in assuming that Y is affine, in which case W is an irreducible component of $V(f_1, ..., f_c)$ as in Proposition 4.1., and Z is contained in an irreducible component $Z' \subset V(\Phi^*(f_1), ..., \Phi^*(f_c))$ of codimension $\leq c$ (also by the Proposition).

But $W = \overline{\Phi(Z)} \subset \overline{\Phi(Z')} \subset V(f_1, ..., f_c)$ and since W is an irreducible component of $V(f_1, ..., f_c)$ it follows that $\overline{\Phi(Z')} = W$. Finally, since Z is a component of $\Phi^{-1}(W)$ and $Z' \subset \Phi^{-1}(W)$, it follows that $Z = Z'$ and the Corollary follows. □

We can eliminate all the components of larger than the "expected" dimension by passing to an open subset of Y:

Theorem 4.3. In the setting of Corollary 4.2, there is a non-empty subset $U \subset Y$ such that:

(i) $U \subset \Phi(X)$ and

(ii) If $W \subset Y$ is a closed subvariety that intersects U, then each irreducible component $Z \subset \Phi^{-1}(W)$ that intersects $\Phi^{-1}(U)$ satisfies:

$$\dim(Z) = \dim(W) + r$$

Proof. As in the Corollary, we may as well assume that Y is affine. We may also assume that X is affine since if the Theorem holds for each of the restrictions $\Phi|_{V_i} : V_i \to X$ for an open affine cover $X = \cup V_i$, with $\Phi|_{V_i}(V_i) \subset U_i$, then the Theorem holds for X itself, with $\Phi(X) \subset \cap U_i$.

So we assume X and Y are affine, and we consider the injective map:

$$f = \Phi^* : k[Y] \to k[X]$$

Let $K = k(Y)$, and apply Noether Normalization to the K-algebra:

$$A = k[X] \otimes_{k[Y]} k(Y) \subset k(X)$$

which is an integral domain that is a finitely generated $k(Y)$-algebra with fraction field $k(X)$, which has transcendence degree r over $k(Y)$.

Thus, by Noether Normalization, there is an integral homomorphism:

$$(*)\ k(Y)[x_1, ..., x_r] \subset A$$

and may also assume, clearing denominators of the $x_i \in A$ by multiplying by suitable elements of the coefficient field $k(Y)$, that each $x_i \in k[X] \subset A$. Consider now the inclusion:

$$k[Y][x_1, ..., x_r] \subset k[X]$$

This may not be integral, but it follows from integrality of $(*)$ that each $\phi \in k[X]$ is the root of a monic polynomial equation:

$$x^n + f_1(x_1, ..., x_r)x^{n-1} + \cdots + f_n(x_1, ..., x_r) = 0$$

such that the f_i are polynomials with coefficients in $k(Y)$. Therefore each ϕ **is** integral over $k[Y][h^{-1}][x_1, ..., x_r]$ for a common denominator h of all the coefficients of the polynomials $f_1, ..., f_n$. Moreover, if we let $\phi_1, ..., \phi_m \in k[X]$ be **generators** of $k[X]$ as an algebra over $k[Y]$ and h_i be the common denominator of polynomials attached to each ϕ_i, then the inclusion:

$$k[Y][h^{-1}][x_1, ..., x_r] \subset k[X][h^{-1}]$$

is an integral extension for $h = h_1 \cdots h_m$. This follows from the fact that sums and products of integral elements over a subring are integral.

Now, we take $U = U_h \subset Y$, and we claim that this choice of U satisfies the Theorem. If we let $V = \Phi^{-1}(U)$, then:

$$\Phi|_V : V \stackrel{\Psi}{\to} U \times k^r \to U$$

is a finite map Ψ followed by a projection, hence it is surjective, proving (i). And if $W \subset Y$ intersects U and $Z \subset \Phi^{-1}(W)$ is a component that intersects V, then let $Z' = Z \cap V$ and $W' = W \cap U$, and note that Z' is finite over $\Psi(Z') \subset W' \times k^r$, so $\dim(Z) \leq \dim(W) + r$. But the other inequality was proved in the Corollary. $\square$

Definition 4.7. A dominant rational map $\Phi : X - - > Y$ of varieties is **birational** if it is associated to an **isomorphism** $k(Y) \cong k(X)$ of fields of rational functions.

Corollary 4.3. Suppose $\Phi : X - - > Y$ is a dominant rational map of varieties. Then there is a non-empty open subset $U \subset Y$ for which the restricted map $\Phi|_V : V = \Phi^{-1}(U) \to U$ is an isomorphism.

Proof. We may restrict to the domain of X to assume that Φ is a regular map. Next, we may restrict to an open affine subset $U \subset Y$ and assume that Y is an affine variety. Using the Theorem, we may assume that X is affine.

Namely, let $U \subset \Phi(X)$ be an open affine satisfying the conditions of the Theorem and let $V \subset \Phi^{-1}(U)$ be an open affine. Then every component $Z \subset X - V$ has smaller dimension than $\dim(X) = \dim(Y)$, and so its image $\overline{\Phi(Z)} \subset Y$ also has smaller dimension. Thus $X - V$ does not dominate Y, and there is a function $f \in k[Y]$ such that $\Phi(X - V) \subset V(f) \subset Y$. Then $U_f \subset U$ is the image of $V_{\Phi^* f} = \Phi^{-1}(U_f)$, which is affine. Thus we may assume that both X and Y are affine.

Now the proof is straightforward. We have the injective map:

$$\Phi^* : k[Y] \to k[X]$$

that induces the isomorphism of fields. Let $x_1, ..., x_n$ generate $k[X]$, and regarding them as elements of $k(Y)$, we may write $x_i = y_i/g$ for $y_i \in k[Y]$ and a common denominator $g \in k[Y]$. But then:

$$k[Y][g^{-1}] \to k[X][g^{-1}]$$

is an isomorphism, and this corresponds to the restrictions of Φ to:

$$\Phi_{V_g} : V_g \to U_g$$

$\square$