6130 Lecture 3

More on Algebraic Sets

· Zariski Topology on k^n

Claim: Let $X \subset k^n$ be closed $\iff X = X(I)$ for some $I \subset k[x_1, \ldots, x_n]$

Then this defines a topology.

Pf: Need to show:

(1) $\emptyset, k^n$ are closed, (2) $\bigcap_\lambda X_\lambda$ is closed

(3) $X_1 \cup X_2$ is closed

(1) $\phi = X(\langle 1 \rangle)$

$$k^n = X(\langle 0 \rangle)$$

(2) If $X_\lambda = X(I_\lambda)$, then

$$\bigcap X_\lambda = X(\sum I_\lambda)$$

$$\sum I_\lambda = \{ f_{\lambda_1} + \cdots + f_{\lambda_m} \}$$

$$f_{\lambda_i} \in I_{\lambda_i} \quad \square$$

(3) If $X_1 = X(I_1)$, $X_2 = X(I_2)$, then $X_1 \cup X_2 = X(I_1 \cdot I_2)$

Example: $(n=1)$ $\quad k^1$

Polynomial ring: $k[x_1]$

Ideals: $I = \langle f \rangle$

$$= \{ \underbrace{(x-r_1)\cdots(x-r_m)}_{\text{roots of } f} \cdot \underbrace{g}_{\text{no roots}} \}$$

$$X(I) = \{ r_1, \dots, r_m \} \subset k$$

Alg subsets of k = finite sets of points.

$X(x_1^2+1) = \emptyset$ in $\mathbb{R}^1$ $\quad (X(x_1) = \{0\})$

Scheme: 1

$k = \mathbb{R}$

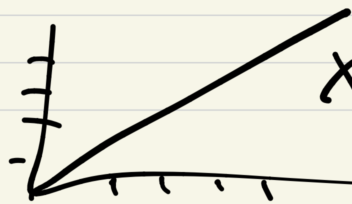
$X(x^2+1)$ has no $\mathbb{R}$-points

$\{i, -i\}$ // has two $\mathbb{C}$-pts.

$k = \mathbb{Q}$ $\quad X(x^2+1)$

$\overline{\mathbb{A}}$ $\quad /\mathbb{Q}$

Ex: k^2 does not have the product topology



$X(x_1 - x_2)$

Feature of Zariski Top:

Descending chains of closed sets stabilize

$$X_1 \supseteq X_2 \supseteq \cdots$$

$$X_n = X_{n+1} = \cdots \quad \text{for some } n.$$

(Reflects the fact that $k[x_1, \dots, x_n]$ is Noetherian)

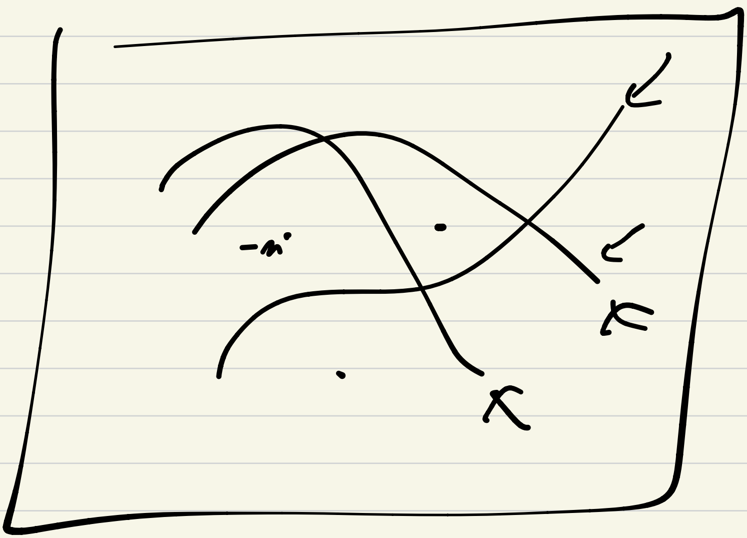
[E.g. $\mathbb{R}^1$ (ordinary top.) is not Noetherian!

$X_i = [-\frac{1}{i}, \frac{1}{i}]$ $\quad \cap X_i = \{0\}$]

$$I(x_1) \subseteq I(x_2) \subseteq \ldots$$

$$X(I(x_n)) = X(I(x_{n+1})) = \ldots$$

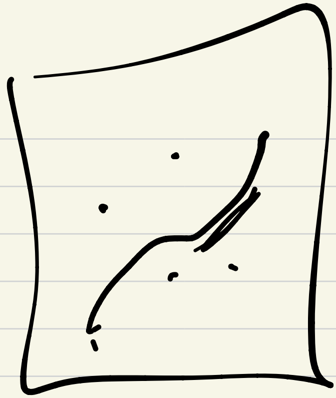
$$X(I(X(I))) = X(I) \; \underline{(!)}$$



$$\underline{X_1} \supseteq \ldots \supseteq$$

Points $a \in k^n$

$= (a_1, \dots, a_n)$

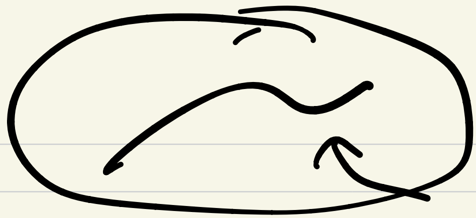


are closed

$$a = X((x_1 - a_1, \dots, x_n - a_n))$$

Towards Hilbert Nullstellensatz:

Setting:



$X = X(P)$

$P \subset k[x_1, \dots, x_n]$ prime

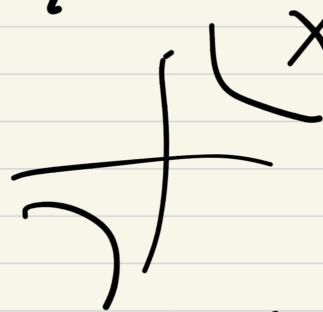
$$k[X] := k[x_1, \dots, x_n] / P$$

$k(X)$ field of fractions

NN: If $d = \operatorname{trdeg}_k k(X)$ and k is infinite, then $\exists$ $y_1 = \sum a_{i,1} x_i, \dots, y_d = \sum a_{i,d} x_i$ $\,/\, a_{ij} \in k$

s.t. y_i are alg. independent and $k[y_1, \dots, y_d] \subset k[X]$ is finite.

Example: Need to be careful choosing y's.



$k[x_1, x_2]$

$P = \langle x_1 x_2 - 1 \rangle$

$k[X] = k[x_1, x_2] / P \simeq k[x_1, x_1^{-1}]$

Rmk: $k[x] \subset k[x_1, x_1^{-1}]$ noeth

is **not** a finite module

$k[x_1] \subseteq x_1^{-1} k[x_1] \subseteq x_1^{-2} k[x_1] \subseteq \dots$

On the other hand: ✓

$$Y_1 = X_1 + X_2 = X_1 + X_1^{-1}$$

$$k[X_1 + X_1^{-1}] \subseteq k[X_1, X_1^{-1}]$$

is generated by

$1, X_1$

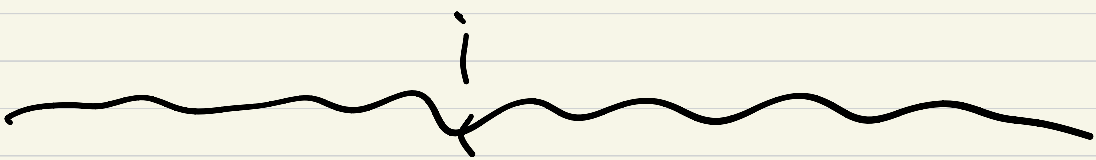
e.g.

$$X_1^{-n} = 1 \cdot f(X_1 + X_1^{-1}) + X_1 \cdot g(X_1 + X_1^{-1})$$

$$(X_1 + X_1^{-1})^n = X_1^{-n} + \text{h.o.}$$

$$\begin{cases} x_1^{-1} = 1\cdot(x_1 + x_1^{-1}) - x_1 \cdot 1 \\ x_1 = (x_1 + x_1^{-1}) - (x_1^{-1}) \end{cases}$$

$$x_2^{-1} = 1\cdot(x_1 + x_1^{-1})^2 - x_1(x_1 - x_1^{-1})$$



Pf: By induction on n

$P \subseteq k[x_1, \dots, x_n]$

If $n = d$, then $P = 0$

and $y_i = x_i$

$k[y_1, \dots, y_d]$

$= k[x_1, \dots, x_d]$

Suppose $n > d$

$$k[x] = k[x_1, \dots, x_n] / P \neq 0$$

Suppose $\exists f \in P$ s.t. $x_1 x_2 - 1$

& $f = a x_n^m + \text{(lower order in } x_n\text{)}$

Then $1, x_n, \dots, x_n^{m-1}$ generate $k[x]$ as a module/

$1, x_n, \dots, x_n^{m-1}$ generate

$k[X]$ as a module over

$= k[x_1, \dots, x_{n-1}] / P \cap k[x_1, \dots, x_{n-1}]$.

$k[Y]$

$\operatorname{tr\,deg}(k(X)) = \operatorname{tr\,deg} k(Y)$

How to arrange:

$f = a x_n^m + (\text{lower order})$

$y_n = \sum a_{i,n} x_i$

$$f = \overbrace{f_{km}}^{\text{highest homo. term}} + \cdots$$

$$f_k = \sum b_I x^I$$

$$|I| = km$$

$$= g\left(\frac{a_{ijn}}{x_0}\right) Y_n^m + \cdots$$

a_{ijn} can be found if k is infinite.

Use to prove Null!