Algebraic Geometry I (Math 6130)

Utah/Fall 2020

4. Projective Varieties.

A projective variety over k is obtained from a $\mathbb{Z}$**-graded** k-algebra domain $A_\bullet$ (via the functor *maxproj*) analogously to the realization of an affine variety from an k-algebra (ungraded) domain A (via the functor *maxspec*). The key difference is that unlike the affine case, in which the domain is recovered from the regular functions, the only regular functions on a projective variety are the constants.

Definition 4.1. As a set, projective space $\mathbb{P}^n_k$ is the locus of lines through $0 \in k^{n+1}$.

Definition 4.2. The polynomial ring graded by degree:

$$S_\bullet = \bigoplus_{d=0}^{\infty} k[x_0, ..., x_n]_d \text{ is defined by}$$

$$S_d = \left\{ \sum_{|I|=d} c_I x_I \mid x_I = x_0^{i_0} \cdots x_n^{i_n}, c_I \in k \right\}$$

i.e. S_d is the vector space of *homogeneous polynomials of degree* d, with:

$$S_d \cdot S_e \subset S_{d+e}$$

Definition 4.3. An ideal $I \subset S_\bullet$ is *homogeneous* if:

$$I = \bigoplus_{d=0}^{\infty} I \cap k[x_0, ..., x_n]_d, \text{ and in that case we let } I_d = I \cap k[x_0, ..., x_n]_d$$

i.e. I is generated by (finitely many!) homogeneous polynomials, so that

$$f = f_0 + \cdots + f_d \in I \ \Leftrightarrow f_e \in I_e \text{ for all } e$$

The quotient by a homogeneous ideal is a graded ring:

$$S_\bullet/I = A_\bullet \text{ with } A_d = S_d/I_d \text{ and } A_d \cdot A_e \subset A_{d+e}$$

Example. (a) The *irrelevant* homogeneous maximal ideal in $S_\bullet$ is:

$$S_+ = \bigoplus_{d=1}^{\infty} k[x_0, ..., x_n]_d = \langle x_0, ..., x_n \rangle$$

This ideal contains **all** homogeneous ideals in $S_\bullet$ other than the ideal $\langle 1 \rangle$.

(b) If $X \subset \mathbb{P}^n_k$, then the *affine cone over* X is:

$$C(X) = \{(a_0, ..., a_n) \in k^{n+1} \mid k \cdot (a_0, ..., a_n) \in X\} \cup \{(0, ..., 0)\}$$

The ideal $I(X) := I(C(X)) \subset S_\bullet$ is a homogeneous ideal (if k is infinite), and:

$$k[X]_\bullet = k[x_0, ..., x_n]_\bullet/I \text{ is the quotient ring}$$

(with this convention, $I(\emptyset) = S_+$, though one could argue for $I(\emptyset) = \langle 1 \rangle$)

(c) For a homogeneous ideal $I \subset S_+$,

$$X(I) = C(X) \subset k^{n+1} \text{ is an affine cone over some } X \subset \mathbb{P}^n_k$$

and we let $X := X(I) \subset \mathbb{P}^n_k$ be the associated algebraic subset of $\mathbb{P}^n_k$.

This sets up a version of the Nullstellensatz for radical homogeneous ideals:

The Projective Nullstellensatz. The radical homogeneous ideals $I \subset S_+$ are in bijection with the algebraic subsets $X = X(I) \subset \mathbb{P}^n_k$ via the mappings X and I, with the prime ideals corresponding to irreducible algebraic sets and the maximal prime ideals properly contained in S_+ corresponding to the points $x \in \mathbb{P}^n_k$ via:

$$m_x = \langle a_j x_i - a_i x_j \rangle \text{ for } x = k \cdot (a_0, ..., a_n)$$

Proof. This follows from the ordinary Nullstellensatz applied to affine cones and the fact that $\mathrm{rad}(I)$ is a homogeneous ideal when I is a homogeneous ideal.

Projective Coordinates. We will write $x \in \mathbb{P}^n_k$ in coordinates as the ratio:

$$(a_0 : \cdots : a_n)$$

with the understanding that $(a_0 : \cdots : a_n) = (\lambda a_0 : \cdots : \lambda a_n)$ for $\lambda \in k^*$.

Remark. If $F \in S_d$ is homogeneous of degree d, then:

$$F(\lambda a_0 : .... : \lambda a_n) = \lambda^d F(a_0 : \cdots : a_n)$$

so although the *value* $F(x)$ is not well-defined, it does make sense to say $F(x) = 0$. When F is not homogeneous, even this statement is not well-defined.

Example. In the projective space $\mathbb{P}^{n^2-1}_k$ of $n \times n$ matrices,

$$X(\Delta) \text{ is the locus (hypersurface) of singular matrices}$$

where $\Delta \in S_n$ is the determinant polynomial. The complement is $\mathrm{PGL}(n, k)$.

The following Lemma is useful.

Lemma 4.4. For a homogeneous ideal $I \subset S_\bullet$,

$$X(I) = \emptyset \Leftrightarrow S_+ \subseteq \mathrm{rad}(I) \ \Leftrightarrow S_d \subset X(I) \text{ for some } d$$

Proof. The first equivalence is immediate, and if $S_+ \subseteq \mathrm{rad}(I)$, then

$$x_i^{d_i} \in I \text{ for some } d_0, ..., d_n$$

and then $S_d \subset I$ for $d > (d_0 + \cdots + d_n) - n$. The converse is clear. □

We now enlarge our stable of $\mathbb{Z}$-graded k-algebra domains to include:

$$k[X]_\bullet = S_\bullet/P \text{ for homogeneous prime ideals } P \subset S_+$$

the homogeneous coordinate rings of irreducible subsets of $\mathbb{P}^n_k$. These rings are:

- $\mathbb{Z}$-graded k-algebra integral domains, with $k[X]_0 = k$
- finitely generated **in degree one** by a basis $x_1, ..., x_n$ of $k[X]_1$.

We now construct a prevariety $(X, \mathcal{O}_X)$ out of each such graded k-algebra $A_\bullet$.

The Set X is the collection of maximal prime ideals $m_x \subset A_+$.

The Topology is the Zariski topology, in which the algebraic sets:

$$X(I) = \{m_x \mid I \subset m_x\}$$

are the closed sets, for (radical) homogeneous ideals $I \subset A_+$.

The Field of Rational Functions is:

$$k(X) = \left\{ \frac{F}{G} \mid F, G \in A_d \ \text{ and } G \neq 0 \right\} \subset k(A)$$

This is a subfield of $k(A)$. The elements of $k(X)$ are homogeneous of degree zero, which makes them (rational) **functions** on X.

Concretely, a choice of basis $x_0, ..., x_n$ of A_1 identifies $A_\bullet = k[x_0, ..., x_n]/P$ and:

$$\text{maxproj}(A_\bullet) = X = X(P) \subset \mathbb{P}^n_k$$

This is an irreducible Zariski topological space by the Projective Nullstellensatz. For $x = (a_0 : ... : a_n) \in X$, and $\phi \in k(X)$,

$$\phi(a_0, ..., a_n) = \frac{F(a_0, ..., a_n)}{G(a_0, ..., a_n)} = \frac{\lambda^d F(a_0, ..., a_n)}{\lambda^d G(a_0, ..., a_n)} = \phi(\lambda a_0, ..., \lambda a_n)$$

is well-defined, provided that $G(a_0, ..., a_n) \neq 0$. More abstractly,

Definition 4.5. A rational function $\phi \in k(X)$ is *regular* at $x \in X$ if

$$\phi = \frac{F}{G} \text{ with } G \notin m_x$$

The rational functions that are regular at $x \in X$ are elements of $A_{(m_x)} \subset k(X)$, a local ring with residue field k, in which the value $\phi(x)$ is taken. The assignment:

$$\mathcal{O}_X(U) = \{\phi \in k(X) \mid \phi \text{ is regular at all points of } U\}$$

defines the sheaf $\mathcal{O}_X$ and the sheaved (Noetherian, irreducible) space maxproj$(A_\bullet)$.

In contrast to Proposition 2.7, we have:

Proposition 4.6. $\mathcal{O}_X(X) = k$ when $(X, \mathcal{O}_X) = \text{maxproj}(A_\bullet)$.

Proof. Let $\phi \in \mathcal{O}_X(X)$ and let $I = \langle G \in A_d \mid G\phi \in A_d \rangle$ be the homogeneous ideal of denominators of I. By assumption $X(I)$ is empty, and if we could conclude (as in the affine case) that $1 \in I$, we'd have $\phi \in A_0 = k$. Instead, we have:

$$A_d \subset I \text{ for some } d \text{ (Lemma 4.4)}$$

In other words, $G\phi \in A_d$ for *all* $G \in A_d$. This has the odd consequence that:

$$G\phi^2 = (G\phi)\phi \in A_d,\ G\phi^3 = (G\phi^2)\phi \in A_d, \text{etc}$$

which gives an increasing chain of submodules:

$$A_\bullet \subset A_\bullet + \phi A_\bullet \subset A_\bullet + \phi A_\bullet + \phi^2 A_\bullet \subset \cdots \subset G^{-1} A_\bullet$$

of a principal graded A-module. Since $A_\bullet$ is Noetherian, the chain stabilizes, and:

$$\phi^n = f_0 + f_1\phi + \cdots + f_{n-1}\phi^{n-1} \text{ for elements } f_i \in A_\bullet$$

In degree 0, this is an identity $\phi^n = c_0 + c_1\phi + \cdots + c_{n-1}\phi^{n-1}$ with coefficients in $k = A_0$, and then since $k = \overline{k}$, it follows that $\phi \in k$, as desired. □

So X isn't affine (unless it is a point). But it is covered by affine varieties:

Proposition 4.7. Each sheaved space $(X, \mathcal{O}_X) = \text{maxproj}(A_\bullet)$ is a prevariety.

Proof. Let $G \in A_d$ be a non-zero element of positive degree d. Then

$$A_{(G)} = \left\{ \frac{F}{G^m} \mid \deg(F) = md \right\} \subset k(X)$$

is a k-algebra domain, generated by y_i/G, where y_i are a basis for A_d. Moreover,

$$k(A_{(G)}) = k(X)$$

and $(U_G, \mathcal{O}_X|_{U_G})$ is isomorphic to maxspec$(A_{(G)})$, where $U_G = X - X(G)$. In this case, we can conclude that G^m is in the ideal of denominators of each $\phi \in \mathcal{O}_X(U_G)$ by the Projective Nullstellensatz, as in Proposition 2.7.

□

Example. The open cover of $\mathbb{P}^n_k$ by $n+1$ affine spaces $U_0, ..., U_n$.

For each of the coordinate functions $x_0, ..., x_n \in k[x_0, ..., x_n]_1$,

$$U_{x_i} = \text{maxspec}(k[x_0, ..., x_n]_{(x_i)}) = \text{maxspec}(k[\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}])$$

is the affine n space of points:

$$U_{x_i} = \{(a_0 : ... : a_n) \mid a_i \neq 0\} = \{(\frac{a_0}{a_i}, ..., 1, ..., \frac{a_n}{a_i})\}$$

Notice in passing that, $\text{PGL}(n, k) = U_\Delta$ is an affine variety, by this Proposition.

A morphism from a prevariety X to affine space $\mathbb{A}^n_k$ is given by regular functions:

$$g_1 = f^*(x_1), ..., g_n = f^*(x_n) \in \mathcal{O}_X(X)$$

via $f(x) = (g_1(x), ..., g_n(x))$. In particular, the only morphisms from a projective prevariety (or any prevariety with $\mathcal{O}_X(X) = k$) to $\mathbb{A}^n_k$ are the constant maps.

But what about morphisms from X to $\mathbb{P}^n_k$? Is there a way to characterize these? The key is *rational functions*. Each prevariety X has its rational function field:

$$k(X) = \lim_{\rightarrow} \mathcal{O}_X(U)$$

When $X = \text{maxspec}(A)$ this is $k(A)$ and when $X = \text{maxproj}(A_\bullet)$, it is $k(X)$. Moreover, if $U \subset X$ is any open subset, then $k(U) = k(X)$.

Definition 4.8. Rational functions $\phi_0, ..., \phi_n \in k(X)$ determine a **rational map**:

$$f : X - - > \mathbb{P}^n_k;\ f(x) = (\phi_0(x) : \cdots : \phi_n(x))$$

The domain of the rational map f is larger than one might expect, since:

$$(\phi_0, ..., \phi_n) \text{ and } (\phi \cdot \phi_0, ..., \phi \cdot \phi_n)$$

determine the same rational map to $\mathbb{P}^n_k$ whenever $\phi \in k(X)^*$. This means that one may be able to expand the domain not just by different forms of $\phi_i = F_i/G_i$, but also by multiplying by convenient rational functions ϕ.

Example. (a) The rational projection map $\pi : \mathbb{P}^2_k - - > \mathbb{P}^1_k$ given by:

$$(\frac{x_1}{x_0} : \frac{x_2}{x_0}) = (1 : \frac{x_2}{x_1}) = (\frac{x_1}{x_2} : 1)$$

is well-defined on the open set $\mathbb{P}^2_k - \{(1 : 0 : 0)\}$ but it cannot be extended further. When restricted to the projective line $X(x_1) \subset \mathbb{P}^2_k$, we get $\pi(a_0 : 0 : a_1) = (0 : 1)$ and when restricted to $X(x_2)$, we get $\pi(a_0 : a_1 : 0) = (0 : 1)$, so there is no way to give a value to $\pi(1 : 0 : 0)$ to extend π to a continuous map. In fact, when restricted to each line through $(1 : 0 : 0)$, the projection map is a **different** constant.

(b) When π is restricted to the conic $C = X(x_1^2 - x_0x_2) \subset \mathbb{P}^2_k$, however:

$$\pi|_C = (1 : \frac{x_2}{x_1}) = (\frac{x_1}{x_2} : 1) = (1 : \frac{x_1}{x_0})$$

with the last form of the map coming from the identity $x_2/x_1 = x_1/x_0$ in $k(C)$. Moreover, this rational map, defined everywhere, inverts $i : \mathbb{P}^1_k \to C$ given by:

$$i = (1 : \frac{x_1}{x_0} : (\frac{x_1}{x_0})^2) = ((\frac{x_0}{x_1})^2 : \frac{x_0}{x_1} : 1)$$

Proposition 4.9. A morphism $f : (X, \mathcal{O}_X) \to \mathbb{P}^n_k$ in the category of sheaved spaces is the same as a rational map that is defined at all points of X.

Proof. We use the open cover of $\mathbb{P}^n_k$ by affine spaces in the Example above. Specifying a morphism $f : X \to \mathbb{P}^n_k$ is the same as specifying morphisms:

$$f_i : W_i \to U_i = \mathbb{A}^n_k \text{ for an open cover } W_i \subset X$$

with the property that $f_i = f_j$ as maps from $W_i \cap W_j$ to $\mathbb{P}^n_k$. Focusing on one i,

$$f_i^*(\frac{x_j}{x_i}) = \phi_i \in \mathcal{O}_{W_i}(W_i) \subset k(X)$$

gives the set $(\phi_0, ..., \phi_n)$ with $\phi_i = 1$ exhibiting f as a rational map. The agreement on the overlap corresponds to replacing each ϕ_i by $\phi \cdot \phi_i$ for $\phi = f_j^*(\frac{x_i}{x_j})$ □

Corollary 4.10. $\mathbb{P}^1_k$ and C from Example (b) above are **isomorphic** prevarieties.

On the other hand, these two projective prevarieties come from the graded rings:

$$A_\bullet = k[x_0, x_1]_\bullet \text{ and } A_{2\bullet} = k[x_0^2, x_0x_1, x_2^2]_\bullet$$

Exercise. maxproj$(A_\bullet)$ and maxproj$(A_{d\bullet})$ are isomorphic prevarieties for all $d > 0$.

Proposition 4.11. Products of projective prevarieties are projective.

Proof. It suffices to prove that $\mathbb{P}^n_k \times \mathbb{P}^m_k$ is a projective prevariety, i.e. to locate this prevariety as a closed, irreducible subset of some $\mathbb{P}^r_k$. Here it is:

$$X = \{\text{rank one } m \times n \text{ matrices}\} \subset \mathbb{P}_k^{(n+1)(m+1)-1}$$

with projective coordinates (a_{ij}) for $i = 0, ..., n$ and $j = 0, ..., m$ and

$$X = X(x_{ij}x_{kl} - x_{il}x_{jk}) \text{ (the vanishing of the two by two minors)}$$

Then X is set-theoretically equal to $\mathbb{P}^n_k \times \mathbb{P}^m_k$ via the *Segre embedding*

$$((a_0 : ... : a_n), (b_0 : ... : b_m)) \mapsto (a_ib_j)$$

and the Cartesian projections are realized by restricting the rational projections:

$$\pi_{\mathbb{P}^n_k} = (x_{10}/x_{ij} : x_{20}/x_{ij} : ... : x_{n0}/x_{ij}) \text{ and } \pi_{\mathbb{P}^m_k} = (x_{01}/x_{ij} : \cdots : x_{0m}/x_{ij})$$

to X (for any choice of x_{ij}), where they are defined everywhere, hence morphisms. On each of the open affines $U_i \times U_j = \mathbb{A}^n_k \times \mathbb{A}^m_k$, this agrees with the product of affine varieties, and so $(X, \pi_{\mathbb{P}^n}, \pi_{\mathbb{P}^m_k})$ is the universal triple. □

Corollary 4.12. Projective prevarieties are varieties.

Proof. The diagonal in $\mathbb{P}^n_k \times \mathbb{P}^n_k$ is the closed subset $X(\{x_{ij} - x_{ji}\}) \subset X$.

It follows that quasi-projective prevarieties $U \subset$ maxproj$(A_\bullet)$ are also varieties.

This choice of an arbitrary x_{ij} in the proof of Proposition 4.11 points to a useful way to think about morphisms from a projective variety X to $\mathbb{P}^n_k$. If $\phi_0, ..., \phi_n$ are rational functions defining a morphism ϕ, then we may choose $G \in A_d$ for some (large) d so that $G\phi_i = F_i \in A_d$ for all i. We may then write f as:

$$f(x) = (F_0(x) : \cdots : F_n(x))$$

and although the values of each $F_i(x)$ individually do not make sense, the ratio does give a well-defined point of projective space, provided that some $F_i(x) \neq 0$. Thus, from this point of view, the projection from $(1 : 0 : 0)$:

$$\pi : \mathbb{P}^2_k - - > \mathbb{P}^1_k \text{ can be written as } \pi(x_0 : x_1 : x_2) = (x_1 : x_2)$$

and the isomorphism from $\mathbb{P}^1_k$ to the conic C can be written as:

$$i : \mathbb{P}^1_k \to \mathbb{P}^2_k;\ i(x_0 : x_1) = (x_0^2 : x_0x_1 : x_1^2)$$

We finish this section with the "completion" of an affine variety. Let

$$A = k[x_1, ..., x_n]/P \text{ with } X = X(P) \subset \mathbb{A}^n_k$$

Then we may *homogenize* the ideal P by homogenizing its elements:

$$P_{hom} = \langle f_{hom} = f(x_1/x_0, ..., x_n/x_0) \cdot x_0^d \rangle | \; f \in P, d = \deg(f) \rangle \subset k[x_0, ..., x_n]_\bullet$$

into generators of P_{hom}. This is a homogeneous prime ideal defining:

$$Y = X(P_{hom}) \subset \mathbb{P}^n_k \text{ satisfying } Y \cap U_0 = X$$

This is the Zariski closure of $Y_0 m X \subset U_0$ as a subset of $\mathbb{P}^n$. The main point is that this closure has an open cover by affine varieties $Y_i = Y \cap U_i$ for all the other open affine space subsets $U_i \subset \mathbb{P}^n$, allowing us to place each of the points in the closure of X in the *interior* of an open affine subvariety of Y.

Example. By this prescription, the closure of the affine curve:

$$X = X(x_2^2 - (x_1^3 + Ax_1 + B)) \subset \mathbb{A}^2_k$$

in the projective plane $\mathbb{P}^2_k$ is:

$$E = X(x_0 x_2^2 - (x_1^3 + Ax_0^2 x_2 + Bx_0^3)) \subset \mathbb{P}^2_k$$

which is obtained from X by adding the single point $(0:0:1) = E \cap X(x_0)$.

The two other affine spaces $U_1, U_2 \subset \mathbb{P}^2_k$ intersect E in affine curves:

$$X_1 = X(x_0 x_2^2 - (1 + Ax_0 x_2^2 + Bx_0^3)) \text{ and } X_2 = X(x_1 - (x_1^3 + Ax_0^2 + Bx_0^3))$$

and it is in X_2 that we may study the elliptic curve "near" the extra point.

Assignment 4.

1. Prove that the projection: $\pi(x_0 : ... : x_n) = (x_0 : ... : x_m)$ is not defined at the points of $\Lambda = X(\langle x_{m+1}, ..., x_n \rangle)$. (a) Show that this is the case by finding:

$$\overline{\pi^{-1}(a_0 : ... : a_m)} \subset \mathbb{P}^n_k - \Lambda \text{ for each point } (a_0 : ... : a_m) \in \mathbb{P}^m_k$$

This is called the linear projection $\pi_\Lambda : \mathbb{P}^n_k - - > \mathbb{P}^m_k$ **from** $\Lambda \subset \mathbb{P}^n_k$.

(b) If $Q = X(x_0 x_3 - x_1 x_2) \subset \mathbb{P}^3_k$, completely describe the projection:

$$\pi_{(0:0:0:1)}|_Q : Q - - > \mathbb{P}^2_k$$

Does it extend across $(0:0:0:1) \in X(Q)$? (c) On the other hand, describe:

$$\pi_\Lambda|_Q : Q - - > \mathbb{P}^1_k \text{ for } \Lambda = \{(* : * : 0 : 0)\} = X(\langle x_2, x_3 \rangle)$$

and show that this does extend across the points of Λ (as in Proposition 4.11.)

2. The d-uple embedding:

$$f_d : \mathbb{P}^n_k \to \mathbb{P}^{\binom{n+d}{d}-1}$$

is given by $f_d(x_0 : ... : x_n) = (... : x_I : ...)$ over all the multi-indices I of degree d.

(a) If $n = 1$, the image of the d-uple embedding is the *rational normal curve*:

$$C_d = \{(a_0^d : a_0^{d-1} a_1 : \cdots : a_1^d) \mid (a_0 : a_1) \in \mathbb{P}^1_k\}$$

corresponding to multi-indices $(d-i, i)$ generalizing the conic from Corollary 4.10. Show that $I(C_d)$ is generated by the 2×2 minors of the matrix:

$$\begin{bmatrix} x_{(d,0)} & x_{(d-1,1)} & \cdots & x_{(1,d-1)} \\ x_{(d-1,1)} & x_{(d-2,2)} & \cdots & x_{(0,d)} \end{bmatrix}$$

(b) If $d = 2$, the embedding $f_2 : \mathbb{P}^n_k \to \mathbb{P}_k^{\binom{n+2}{2}-1}$ is the **Veronese embedding.** In this case, the monomials of degree 2 are all of the form $x_i x_j$, and f_2 can be thought of as:

$$f_2(a_0 : .... : a_n) = (... : a_i a_j : ...)$$

whose coordinates can be arranged in a symmetric $n+1 \times n+1$ matrix $A = (a_{i,j})$. Show that the image is the rank one locus in symmetric all matrices $(x_{i,j})$, and is therefore cut out by the quadratic equations of the principal 2×2 minors. Work out the explicit quadratic equations for the Veronese embedding of $\mathbb{P}^2$.

(c) In general, arrange the multi-indices in a convenient ordering to show that that d-uple embedding is an isomorphism from $\mathbb{P}^n_k$ to its image via an appropriate inverse projective mapping.

3. The **Grassmannian** $G(m, n)$ is the set of m-planes in k^n (e.g. $G(1, n) = \mathbb{P}^{n-1}_k$). Consider the rational map:

$$\mathbb{P}(\text{Hom}(k^m, k^n)) - - > \mathbb{P}^{\binom{n}{m}-1}$$

given by the $m \times m$ **minors** of a matrix $A \in \text{Hom}(k^m, k^n)$. Work this out explicitly for the case $m = 2$ and $n = 4$ and convince yourself that the image is $X(q) \subset \mathbb{P}^5_k$ for a suitable nonsingular (see Problem 5) quadratic polynomial. The image also can be interpreted as the set of indecomposable alternating tensors:

$$v_1 \wedge \cdots \wedge v_m \text{ in } \wedge^m k^n$$

4. (a) Prove Euler's formula for homogeneous polynomials $F \in k[x_0, ..., x_n]_d$.

$$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = dF$$

(b) The projective tangent plane $T_p(X(F)) \subset \mathbb{P}^n_k$ to $X(F)$ at $p \in X(F)$ is:

$$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i}(p) = 0$$

provided that the gradient $\nabla(F)(p) \neq 0$.

The affine tangent plane to $X(f)$ for $f \in k[x_1, ..., x_n]$ vanishing at $(0, ..., 0)$ is:

$X(f_1)$ where $f = f_1 + f_2 + \cdots + f_d$ are the homogeneous terms of f

Show that if $F(p) = 0$ and $p = (1 : 0 : ... : 0)$, then:

$T_p(X(F)) \cap U_0$ is the affine tangent plane to $X(f) = X(F) \cap U_0$ at $(0, ..., 0)$

and that if $\nabla(F)(p) = 0$, then $f_1 = 0$ for the polynomial $f = F(1, x_1/x_0, ..., x_n/x_0)$.

Thus, $p \in X(F)$ is a singular point (no tangent plane) if and only if $\nabla(F)(p) = 0$. In particular, if $k = \mathbb{C}$ and $\nabla(F)(p) \neq 0$, then $X(F)$ is a complex manifold of dimension n in a Zariski open neighborhood of $p \in X(F)$.

(c) Show that the elliptic curve $X(y^2 - x^3 - Ax - B)$ is non-singular at the "point at infinity" and find its projective tangent line.

5. In the projective plane $\mathbb{P}^2_k$, the simplest singularities are simple nodes and cusps. If $f(x_1, x_2) = f_2 + f_3 + \cdots + f_d$ is singular at $(0, 0)$, then:

$$f_2(x_1, x_2) = (a_1 x_1 - a_2 x_2)(b_1 x_1 - b_2 x_2)$$

(we're assuming $k = \overline{k}$), and then:

(i) $X(F)$ has a simple node at $(1:0:0)$ if $(a_2 : a_1) \neq (b_2 : b_1) \in \mathbb{P}^1$, i.e. if the linear factors of f_2 define different lines through $(0,0)$.

(ii) $X(F)$ has a simple cusp at $(1:0:0)$ if the linear factors of f_2 are dependent (but not zero).

Question. How do we interpret this in terms of the tangent **cone**:

$$\sum_{i,j} x_i x_j \frac{\partial^2 F}{\partial x_i \partial x_j} = 0$$

at $p \in X(F)$ of a singular point of $X(F) \subset \mathbb{P}^2_k$?

5. A homogeneous **quadric** is a quadratic form:

$$q = \sum_{i \le j} c_{i,j} x_i x_j \in k[x_0, ..., x_n]_2$$

which is identified with the symmetric matrix:

$$Q = \begin{bmatrix} c_{0,0} & \frac{1}{2}c_{0,1} & \cdots & \frac{1}{2}c_{0,n} \\ \frac{1}{2}c_{0,1} & c_{1,1} & \cdots & \frac{1}{2}c_{1,n} \\ & & \vdots & \\ \frac{1}{2}c_{0,n} & \frac{1}{2}c_{1,n} & \cdots & c_{n,n} \end{bmatrix}$$

so that

$$q(x_0, ..., x_n) = \vec{x}^T Q \vec{x} \text{ for the column vector } \vec{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Prove that the singular locus of the **quadric hypersurface** $X(q)$ is:

$$\Lambda = \mathbb{P}(\ker(Q)) \subset \mathbb{P}^n_k$$

so that in particular, $X(q)$ is non-singular if and only if $\det(Q) \neq 0$.

Show (diagonalizing the quadric if like) that the projection from Λ realizes $X(q)$ as the inverse image of a nonsingular quadric $X(q_0)$ (closed up to include Λ) under the projection map:

$$\pi_\Lambda : \mathbb{P}^n - - > \mathbb{P}(\mathrm{im}(Q))$$

This is called the **cone over the quadric** $X(q_0) \subset \mathbb{P}(\mathrm{im}(Q))$.

6. Prove that the only automorphisms of $\mathbb{P}^n_k$ (as projective varieties) are the natural (transitive) action of $\mathrm{PGL}(n,k)$ What are the automorphisms of a non-singular quadric $Q \subset \mathbb{P}^n_k$?