Lesson Twelve
Math 6080 (for the Masters Teaching Program), Summer 2020

Euler’s Proof of Euclid’s Theorem. Recall that the harmonic series:

\[ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \]

diverges, which is to say that it eventually surpasses every natural number.

On the other hand, the geometric series of the powers of 1/2 converges:

\[ 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots = 2 \]

For every prime number \( p \), the geometric series of powers of 1/p converges:

\[ 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^n} + \cdots = \frac{1}{1 - \frac{1}{p}} = \frac{p}{p - 1} \]

Now suppose we multiply two of them:

\[ (1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^n} + \cdots) \]

On the one hand:

\[ 2 \left( \frac{3}{3 - 1} \right) = 3 \]

but on the other hand, by distributing the multiplication, we obtain:

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{9} + \frac{1}{8} + \cdots + \frac{1}{2^n 3^n} + \cdots = 3 \]

which is the sum of the reciprocals of every number with only 2 and 3 as factors.

If we do this for all the primes, we get:

\[ (\ast) \text{ harmonic series } = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \cdots \cdot \frac{p}{p - 1} \cdots \]

In particular, there cannot be finitely many prime because the left side diverges!

But Euler gets an even better result. Let’s review some Calculus.

(i) the sum of the harmonic series to \( 1/n \) is trapped between \( \ln(n) \) and \( \ln(n) + 1 \).

We can numerically check this with Python! Thus the harmonic series very slowly diverges, dancing an intimate slow dance with the natural logarithm.

(ii) the Maclaurin power series for \( \ln(1 - x) \) is:

\[ \ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \]

In particular, setting \( x = -1 \), we get:

\[ \ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots \]

which is the alternating harmonic series, whose convergence we can again check numerically with Python. This converges fairly quickly. If \( n \) is odd, then:

\[ 1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{n - 1} < \ln(2) < 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{n} < \ln(2) + \frac{1}{n} \]

because the series is alternating, so the series summed up to \( 1/n \) is trapped between \( \ln(2) \) and \( \ln(2) + 1/n \).
(iii) Now let’s take the natural logarithm of both sides of (\ast) above:

$$\ln(\text{harmonic series}) = \ln \left( \prod_p \left( \frac{1}{1 - \frac{1}{p}} \right) \right) = \sum \ln \left( \frac{1}{1 - \frac{1}{p}} \right)$$

$$= \sum \left( \frac{1}{p} - \frac{1}{2p^2} - \frac{1}{3p^3} - \cdots \right)$$

If we rearrange\* the terms of this infinite sum of infinite sums, we get:

$$\ln(\text{harmonic series}) = \sum \frac{1}{p} + \frac{1}{2} \sum \frac{1}{p^2} + \frac{1}{3} \sum \frac{1}{p^3} + \cdots$$

and all the terms (and their infinite sum!) other than the first term converge.

The consequence of this is:

$$\ln(\text{harmonic series}) < \sum \frac{1}{p} + \text{constant}$$

But \(\ln(\ln(n))\) goes to infinity as \(n\) goes to infinity, so the sum of \(1/p\) diverges!

As noted earlier, this says more than simply that there are infinitely many primes. **Dirichlet’s Theorem** is a variation in which one fixes a “modulus” \(m\) and asks:

For each remainder \(r\) between 0 and \(m - 1\), what “proportion” of the primes satisfy:

\[ p \% m = r \]

For example, suppose \(m = 3\). Then:

0. 3 is divisible by 3.

1. 7, 19, ... satisfy \(p \% 3 = 1\).

2. 2, 5, 11, ... satisfy \(p \% 3 = 2\).

As another example, suppose \(m = 4\). Then:

0. Nothing

1. 5, 17, 29, ... satisfy \(p \% 4 = 1\).

2. 2 satisfies \(p \% 4 = 2\). Nothing else.

3. 3, 7, 11, ... satisfy \(p \% 4 = 3\).

**Dirichlet’s Theorem.** For each fixed modulus \(m\).

(i) If gcd\((m, r) \neq 1\), then at most one prime satisfies \(p \% m = r\).

(ii) For all the remainders \(r\) that do satisfy gcd\((m, r) = 1\), the numbers of primes between 1 and \(n\) satisfying \(p \% m = r\) are approximately the same. In an appropriate sense, the infinitely many primes are evenly distributed among these remainders.

**Remark.** The proof of (i) is easy. If \(p \% m = r\), then:

\[ \gcd(m, r) = \gcd(m, p) = d \]

and so \(d\) divides \(p\). But if \(p\) is prime, then we must have \(d = 1\) or \(d = p\).

(ii) is hard.
Our Challenge. To write Python code to check (ii) numerically.

The Strategy. Use the Sieve of Eratosthenes to create a list of lists.

```python
Dirichlet = []
for r in range(m):
    Dirichlet = Dirichlet + [[]]
```

This creates a list of $m$ empty lists, with $\text{Dirichlet}[r] = []$.

Now we feed into each $\text{Dirichlet}[r]$ all the primes in the Sieve with $p \% m = r$.

Then we compare the values $\text{len}(\text{Dirichlet}[r])$ as $r$ ranges from 0 to $m - 1$ and numerically “see” the even distribution of the primes from Dirichlet’s Theorem.

Exercise. Write the code to do this.