1. **Affine Varieties.** To first approximation, an affine variety is the locus of zeroes (in $\mathbb{C}^n$) of a system of polynomials:

$$f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n]$$

Systems of polynomials are better expressed in terms of the ideals that they generate, and two theorems by Hilbert on ideals are the starting point for an “intrinsic” treatment of affine varieties.

**Theorem (Hilbert Basis):** Every ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ can be generated by finitely many polynomials: $I = \langle f_1, \ldots, f_m \rangle$.

**Definition:** Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a nonconstant polynomial. Then:

$$V(f) := \{ (a_1, \ldots, a_n) \in \mathbb{C}[x_1, \ldots, x_n] \mid f(a_1, \ldots, a_n) = 0 \}$$

is the **hypersurface** in $\mathbb{C}^n$ defined as the zero locus of the polynomial $f$.

**Definition:** Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be an ideal. Then:

$$V(I) := \{ (a_1, \ldots, a_n) \in \mathbb{C}[x_1, \ldots, x_n] \mid f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I \}$$

is the **algebraic subset** of $\mathbb{C}^n$ determined by the polynomials in $I$.

**Corollary 1.1:** An algebraic subset of $\mathbb{C}^n$ is either $\mathbb{C}^n$ itself, or else it is the intersection of finitely many hypersurfaces.

**Theorem (Hilbert Nullstellensatz):** The natural map:

$$m : \mathbb{C}^n \to \{ \text{maximal ideals in } \mathbb{C}[x_1, \ldots, x_n] \}$$

$$m_{(a_1, \ldots, a_n)} := \text{the kernel of “evaluation at } (a_1, \ldots, a_n)\text{”}$$

is a bijection (and note that $m_{(a_1, \ldots, a_n)} = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$).

**Corollary 1.2:** If $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is an ideal, then the restriction:

$$m|_{V(I)} : V(I) \to \{ \text{maximal ideals containing } I \}$$

is a bijection

**Corollary 1.3:** The natural “closure” on ideals in $\mathbb{C}[x_1, \ldots, x_n]$:  

$$I \mapsto \overline{I} := \bigcap_{x \in V(I)} \{ \text{maximal ideals } I \subset m_x \}$$

coincides with the “radicalization:”

$$\overline{I} = \sqrt{I} := \{ f \in \mathbb{C}[x_1, \ldots, x_n] \mid f^N \in I \text{ for some } N > 0 \}$$

In particular, a prime ideal is equal to its closure (= radical).

**Review** the proofs of the Nullstellensatz and Corollary 1.3.
Definition: Let $S \subset \mathbb{C}^n$ be an arbitrary subset. Then:

$$I(S) := \bigcap_{x \in S} m_x$$

is the ideal in $\mathbb{C}[x_1, \ldots, x_n]$ of polynomials that vanish on $S$.

Definition: An algebraic subset $V(I) \subset \mathbb{C}^n$ is irreducible if:

$$V = V_1 \cup V_2 \Rightarrow V = V_1 \text{ or } V = V_2$$

whenever $V_1 = V(I_1)$ and $V_2 = V(I_2)$ are algebraic sets.

Exercise 1.1: If $V(I)$ is an irreducible algebraic set, then:

$$\sqrt{I} = I(V(I))$$

is a prime ideal, and vice versa, $V(P) \subset \mathbb{C}^n$ is an irreducible algebraic set whenever $P \subset \mathbb{C}[x_1, \ldots, x_n]$ is a prime ideal.

Complex affine varieties are the irreducible algebraic sets in $\mathbb{C}^n$ (for some $n$). It is crucial to have an “intrinsic” description of an affine variety (i.e. without reference to the ambient $\mathbb{C}^n$), as a topological space equipped with a sheaf of $\mathbb{C}$-algebras. To this end, suppose:

$$X = V(P) \subset \mathbb{C}^n$$

is an irreducible algebraic set. Then:

- The coordinate ring of $X$ is $\mathbb{C}[X] := \mathbb{C}[x_1, \ldots, x_n]/P$. ($\mathbb{C}[X]$ is an integral domain that is finitely generated as a $\mathbb{C}$-algebra.)
- The field of rational functions on $X$ is the fraction field:

$$\mathbb{C}(X) := \left\{ \frac{f}{g} \mid f, g \in \mathbb{C}[X] \text{ and } g \neq 0 \right\} / \left( \frac{f}{g} \sim \frac{f'}{g'} \text{ if } fg' = f'g \right)$$

($\mathbb{C}(X)$ has finite transcendence degree as a field extension of $\mathbb{C}$.)
- The points of $X$ are in bijection with the maximal ideals in $\mathbb{C}[X]$:

$$\{ x \in X \} \leftrightarrow \{ \text{maximal ideals } m_x \subset \mathbb{C}[X] \}$$

- The Zariski topology on $X$ is generated by “open” sets of the form:

$$U_f := \{ x \in X \mid f \notin m_x \} \subset X \text{ for each } f \in \mathbb{C}[X]$$

($U_f$ is the complement (in $X$) of the hypersurface $V(f) \subset \mathbb{C}^n$)
- The germ of rational functions at each $x \in X$ is:

$$\mathcal{O}_x := \left\{ \frac{f}{g} \mid g \notin m_x \right\} \subset \mathbb{C}(X)$$

($\mathcal{O}_x$ has unique maximal ideal $\tilde{m}_x := \left\{ \frac{f}{g} \mid f \in m_x, g \notin m_x \right\} \subset \mathcal{O}_x$)
Note: I stopped writing the equivalence relation. So sue me.

- The regular functions on each open set $U \subset X$ are:
  \[ \mathcal{O}_X(U) := \bigcap_{x \in U} \mathcal{O}_x \subset \mathbb{C}(X) \]
  
  ($\mathcal{O}_X(U)$ is an integral domain containing $\mathbb{C}[X]$.)

**Exercise 1.2.** (a) Prove that the sets:

\[ U_I := \{ x \in X \mid I \notin m_x \} \]

are always finite unions of the $U_f \subset X$, and that these sets are precisely the open sets of the Zariski topology on $X$.

(b) Prove that

\[ \mathcal{O}_X(U_f) = \left\{ \frac{g}{f^n} \mid n \geq 0, g \in \mathbb{C}[X] \right\} \subset \mathbb{C}(X) \]

In particular, conclude that $\mathcal{O}_X(X) = \mathbb{C}[X]$.

(c) Prove that each $\mathcal{O}_X(U)$ is finitely generated as a $\mathbb{C}$-algebra.

(d) Show that the germ of rational functions $\mathcal{O}_x$ at $x$ satisfies

\[ \mathcal{O}_x = \bigcup_{x \in U} \mathcal{O}_X(U) \]

and is **not** usually finitely generated as a $\mathbb{C}$-algebra.

(e) Prove that

\[ \mathbb{C}(X) = \bigcup_{x \in X} \mathcal{O}_x = \bigcup_{U \subset X} \mathcal{O}_X(U) \]

Note also that $U \subset V \Rightarrow \mathcal{O}_X(V) \subset \mathcal{O}_X(U)$.

**Exercise/Example 1.3:** Describe all the data above for $X = \mathbb{C}^1$ and $\mathbb{C}^2$. In particular, show that:

\[ \mathcal{O}_{\mathbb{C}^2}(\mathbb{C}^2 - \{0\}) = \mathbb{C}[x_1, x_2] = \mathcal{O}_{\mathbb{C}^2}(\mathbb{C}^2) \]

showing that shrinking an open set does not always increase the ring of regular functions.

Next, prove that the Cartesian product of open sets in $\mathbb{C}^1$ is an open set in $\mathbb{C}^2$, but that, for example, the open subset:

\[ U_{x_1 - x_2} \subset \mathbb{C}^2 \text{ (the complement of the diagonal)} \]

does **not** contain any (non-empty) product of open sets in $\mathbb{C}^1$. Thus, in particular, the Zariski topology on $\mathbb{C}^2$ is not the product topology of the two Zariski topologies on $\mathbb{C}^1$. 
2. Abstract Varieties. An abstract variety is a set with a (Zariski) topology and a sheaf of functions that is locally affine and separated. In order to define these terms properly, we need to define them in the context of an appropriate category.

Let \( X \) be a topological space and \( A \) be a commutative ring with 1.

**Definition.** A sheaf \( S \) of \( A \)-valued functions on \( X \) consists of:

(a) A commutative ring with 1, denoted \( S(U) \), consisting of a subring of the ring of functions \( f : U \to A \) for each open \( U \subset X \), such that:

(b) functions in \( S(V) \) restrict to functions in \( S(U) \) if \( U \subseteq V \), and

(c) functions \( f_i \in S(U_i) \) that agree when restricted to intersections \( U_i \cap U_j \) are restrictions of a single well-defined function \( f \in S(\cup U_i) \).

**Examples:**

(a) The sheaf of “totally discontinuous” functions on \( X \). In this sheaf, denoted \( A^\text{disc}_X \), each \( A^\text{disc}_X(U) \) consists of all \( f : U \to A \).

(b) At the other extreme, the constant functions do not form a sheaf. But there is a sheaf, denoted \( A^\text{const}_X \), consisting of functions that are locally constant. In this sheaf, \( A^\text{const}_X(U) \) consists of all the functions \( f : U \to A \) that are constant on each connected component of \( U \).

(c) If \( A \) has a topology (e.g. \( \mathbb{R} \) or \( \mathbb{C} \) with the Euclidean topology), then the functions \( f : U \to A \) that are continuous form a sheaf, often denoted simply by \( C^\infty_X \). This generalizes examples (a) and (b) (Why?)

(d) If \( X \) is a differentiable manifold, then the rings of “infinitely differentiable” functions \( f : U \to \mathbb{R} \) form a sheaf, denoted by \( C^\infty_X \).

(e) The regular functions on an affine variety \( X \) with the Zariski topology form a sheaf of \( \mathbb{C} \)-valued functions, denoted \( \mathcal{O}_X \), which has the unusual property that \( U \subseteq V \) implies \( \mathcal{O}_X(V) \subseteq \mathcal{O}_X(U) \). In other words, the restriction of such functions is an injective map for this sheaf. This is not true of any of the other sheaves discussed here (Why?)

(f) Suppose \( V \subset X \) is an open subset and \( S \) is a sheaf of \( A \)-valued functions on \( X \). Then the induced topology on \( V \), together with the “restricted sheaf” \( S|_V \) defined by: \( S|_V(U) := S(U) \) for all \( U \subseteq V \) is a sheaf of \( A \)-valued functions on \( V \).

**Definition:** A morphism between pairs \( (X, S_X) \) and \( (Y, S_Y) \) consisting of a topological space with a sheaf of \( A \)-valued functions is:

(a) A continuous map \( F : X \to Y \) with the property that:

(b) the pull-back on functions defined by \( F^*(f) := f \circ F \) maps each:

\[
F^* : S_Y(U) \to S_X(F^{-1}(U))
\]
This gives the collection of pairs $(X, S_X)$ the structure of a category.

**Examples:** (a) For any $X$, the identity map defines a morphism:
$$\text{id} : (X, A_X^{\text{disc}}) \to (X, A_X)$$
but not in the opposite direction (unless $X$ is discrete)! Similarly,

(b) The identity map on a differentiable manifold $X$ defines:
$$\text{id} : (X, C_X) \to (X, C_X^\infty)$$
but not in the opposite direction.

(c) If $\iota : V \subset X$ is an open set together with the restricted sheaf $S|_V$ (for any sheaf $S$ of $A$-valued functions), then the inclusion map
$$\iota : (V, S|_V) \to (X, S)$$
is a morphism.

(d) A continuous mapping of differentiable manifolds $F : X \to Y$ is (infinitely) differentiable if and only if it defines a morphism:
$$F : (X, C_X^\infty) \to (Y, C_Y^\infty)$$
i.e. it pulls back (locally) infinitely differentiable functions on $Y$ to (locally) infinitely differentiable functions on $X$.

(e) Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be irreducible algebraic sets, with Zariski topologies and coordinate rings $\mathbb{C}[X]$ and $\mathbb{C}[Y]$. A morphism $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in the category of topological spaces with sheaves of $\mathbb{C}$-valued functions determines a $\mathbb{C}$-algebra homomorphism $F^* : \mathcal{O}_Y(Y) = \mathbb{C}[Y] \to \mathbb{C}[X] = \mathcal{O}_X(X)$, and conversely:

**Proposition 2.1:** Each $\mathbb{C}$-algebra homomorphism $\Phi : \mathbb{C}[Y] \to \mathbb{C}[X]$ comes from a uniquely determined morphism:
$$F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$
in the sense that $\Phi = F^* : \mathbb{C}[Y] \to \mathbb{C}[X]$.

**Proof:** Recall that the points of $X$ are in a natural bijection with the maximal ideals of $\mathbb{C}[X]$. Thus:
$$F(x) = y \iff \Phi^{-1}(m_x) = m_y$$
is well-defined, and it is the only possible map for which $F^* = \Phi$ on (globally defined) regular functions. Moreover,

- $F^{-1}(U_g) = U_{\Phi(g)}$ for all $g \in \mathbb{C}[Y]$, so $F$ is continuous, and
- $F^*(f/g^n) = \Phi(f)/\Phi(g)^n$ shows that $F^* : \mathcal{O}_Y(U_g) \to \mathcal{O}_X(U_{\Phi(g)})$
from which it follows that $F$ is a morphism, as desired.
**Corollary 2.2:** Irreducible algebraic sets, with rational functions:

\((X, \mathcal{O}_X)\) and \((Y, \mathcal{O}_Y)\)

are isomorphic in the category of topological spaces with sheaves of \(\mathbb{C}\)-valued functions if and only if \(\mathbb{C}[X] \cong \mathbb{C}[Y]\) as \(\mathbb{C}\)-algebras.

**Refined Definition (of an affine variety):** A complex affine variety is a topological space \(X\) with sheaf \(\mathcal{S}_X\) of \(\mathbb{C}\)-valued functions that is isomorphic to some \((Y, \mathcal{O}_Y)\), where \(Y\) is an irreducible algebraic set in some \(\mathbb{C}^m\) and \(\mathcal{O}_Y\) is its sheaf of regular functions.

**Important Example:** If \(X = V(\mathcal{P}) \subset \mathbb{C}^n\) and \(f \in \mathbb{C}[x_1, \ldots, x_n]\), then \((U_f, \mathcal{O}_X|_{U_f})\) (for \(U_f \subset X\)) is an affine variety. It is isomorphic to the “affine hyperbola over \(U_f\),” namely

\[Y := V(\langle \mathcal{P}, 1 - fx_{n+1} \rangle) \subset \mathbb{C}^{n+1}\]

which is an irreducible algebraic set with coordinate ring:

\[\mathbb{C}[Y] \cong \mathbb{C}[X][x_{n+1}]/(1 - fx_{n+1}) \cong \mathbb{C}[X][f^{-1}] \subset \mathbb{C}(X)\]

**Definition:** A pair \((X, \mathcal{S}_X)\) consisting of a topological space with sheaf of \(\mathbb{C}\)-algebras is a prevariety if \(X\) is connected and covered by open sets:

\[X = \bigcup_{i=1}^n U_i\]

with the property that each of the pairs \((U_i, \mathcal{S}_X|_{U_i})\) is an affine variety.

**Corollary (of the important example) 2.3:** Every (Zariski) open subset \(U \subset X\) (with sheaf \(\mathcal{O}_X|_U\)) of an affine variety is a prevariety.

**Exercise 2.1:** The open subset \(U = \mathbb{C}^2 - \{(0,0)\} \subset \mathbb{C}^2\) with sheaf \(\mathcal{O}_U = \mathcal{O}_{\mathbb{C}^2}|_U\) is a prevariety but not an affine variety.

**Gluing:** Let \((X, \mathcal{S}_X)\) and \((Y, \mathcal{S}_Y)\) be topological spaces with sheaves of \(A\)-valued functions that have isomorphic open subsets, specifically \(U \subset X\) and \(V \subset Y\) with an isomorphism \(F : (U, \mathcal{S}_X|_U) \sim (V, \mathcal{S}_Y|_V)\)

Then we may “glue \(X\) and \(Y\) along \(F\)” to obtain \((Z, \mathcal{S}_Z)\) defined by:

- As a set,

\[Z = (X \coprod Y)/\sim \text{ where } x \sim F(x) \text{ for each } x \in U\]

with natural inclusion maps \(\iota_X : X \subset Z\) and \(\iota_Y : Y \subset Z\).

- \(W \subset Z\) is open if and only if both \(W \cap X\) and \(W \cap Y\) are open.

- \(\mathcal{S}_Z(W)\) is the set of functions \(f : W \to A\) satisfying:

\[f|_{W \cap X} \in \mathcal{S}_X(W \cap X), \ f|_{W \cap Y} \in \mathcal{S}_Y(W \cap Y)\]

and \(F^*(f|_{W \cap Y}) = f|_{W \cap U}\).
Conversely: Given \((X, S_X)\) and open sets \(U \subset X\) and \(V \subset X\), then \((X, S_X)\) is isomorphic to the topological space with sheaf of \(A\)-valued functions obtained by gluing \((U, S_X|_U)\) to \((V, S_X|_V)\) along the canonical isomorphism \(F : U \cap V \cong V \cap U\).

**Corollary 2.3:** A prevariety is obtained by gluing an affine variety to another affine variety (or prevariety) along non-empty open subsets.

**Two Very Different Examples:** One can glue

\((\mathbb{C}, \mathcal{O}_\mathbb{C})\) to \((\mathbb{C}, \mathcal{O}_\mathbb{C})\)

along \(\mathbb{C}^* = \mathbb{C} - \{0\}\) in two different ways:

(i) Gluing along the identity isomorphism \(\text{id} : \mathbb{C}^* \to \mathbb{C}^*\) produces "the affine line with doubled origin." In the framework of manifolds, this is a simple example of a non-Hausdorff "fake" manifold. Unfortunately, the Zariski topology on a variety is essentially never Hausdorff (since all pairs of nonempty open subsets of an affine variety intersect). Thus, we will have to find another way to eliminate it.

However, there is another interesting automorphism of \(\mathbb{C}^*\). Since:

\[(\mathbb{C}^*, \mathcal{O}_{\mathbb{C}|\mathbb{C}^*}) \cong (X, \mathcal{O}_X)\]

where \(\mathbb{C}[X] \cong \mathbb{C}[x, x^{-1}]\) by the affine hyperbola construction, it follows that the \(\mathbb{C}\)-algebra automorphism:

\[\mathbb{C}[X] \to \mathbb{C}[X]; \quad x \mapsto x^{-1}\]

is associated to an automorphism \(F\) of \((\mathbb{C}^*, \mathcal{O}_{\mathbb{C}^*})\).

(ii) Gluing along \(x \mapsto x^{-1}\) produces the projective line.

(We will explore this in detail later.)

**Definition:** A *product* of objects \(X, Y\) of a category \(\mathcal{C}\) is an object, which we will denote by \(X \times Y\), together with “projection” morphisms \(\pi_X : X \times Y \to X\) and \(\pi_Y : X \times Y \to Y\) with the following:

*Universal property:* To each triple \((Z; F, G)\) consisting of an object \(Z\), with morphisms \(F : Z \to X\) and \(G : Z \to Y\), there is a unique:

\[Z \to X \times Y\]

that commutes with the morphisms to \(X\) and \(Y\).

The product is unique (if it exists) up to a uniquely determined isomorphism. Thus we can get away the leading notation “\(X \times Y\).”

**Example:** The Cartesian product is a product in the category of sets. In the category of topological spaces, the Cartesian product, together with the *product topology*, is a product.
**Lookup:** The tensor product $C[X] \otimes_C C[Y]$ of integral domains that are finitely generated as $C$-algebras is also an integral domain, and also finitely generated as a $C$-algebra.

**Corollary 2.4:** If $X$ and $Y$ are affine varieties, then

$$C[X] \otimes_C C[Y] \cong C[X \times Y]$$

for an affine variety $X \times Y$, which, together with morphisms $\pi_X$ and $\pi_Y$ associated to the inclusions $C[X] \subseteq C[X \times Y]$ and $C[Y] \subseteq C[X \times Y]$, respectively, is the product of $X$ and $Y$ in the category of affine varieties. This is due to the analogous universal property of the tensor product.

**Warning:** Note that $\mathbb{C}^m \times \mathbb{C}^n \cong \mathbb{C}^{m+n}$, but that, as we’ve already noted in an earlier exercise, the Zariski topology on this product is not in general equal to the product topology. This does not contradict the Example above describing the products of topological spaces.

**Definition:** In a category whose objects are topological spaces, whose morphisms are continuous, and in which products exist, an object $X$ is separated if the image of the canonical diagonal map

$$\delta : X \rightarrow X \times X$$

is closed.

**Exercise 2.2:** (a) In the category of topological spaces, prove that $X$ is Hausdorff if and only if it is separated.

(b) Prove that as a set, the product $X \times Y$ of affine varieties is the Cartesian product of $X$ and $Y$ (this is not, however, true of schemes).

**Proposition 2.5:** Affine varieties are separated.

**Proof:** Let $X$ be an affine variety. Via an isomorphism we may assume $X = V(P) \subset \mathbb{C}^n$ with coordinates $x_1, \ldots, x_n$. Then $X \times X \subset \mathbb{C}^{2n}$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$, and $\delta(X) \subset X \times X$ is the closed subset defined by the equations $\{x_i - y_i = 0 \mid i = 1, \ldots, n\}$.

**More Involved Exercises 2.3:**

(a) Prove that products exist in the category of prevarieties.

(b) A quasi-affine variety is, by definition, a pair $(U, O_{X|U})$, where $U \subset X$ is a (non-empty) open subset of an affine variety $X$. Prove that quasi-affine varieties are separated.

(c) Prove that the affine line with the doubled origin is not separated.

(d) Prove that the projective line is separated.

**Definition:** An abstract variety is a separated prevariety.
3. Projective Varieties. To first approximation, a projective variety is the locus of zeroes of a system of *homogeneous* polynomials:

\[ F_1, \ldots, F_m \in \mathbb{C}[x_1, \ldots, x_{n+1}] \]

in projective \(n\)-space. More precisely, a projective variety is an abstract variety that is isomorphic to a variety determined by a homogeneous prime ideal in \(\mathbb{C}[x_1, \ldots, x_{n+1}]\). Projective varieties are *proper*, which is the analogue of “*compact*” in the category of abstract varieties.

**Projective \(n\)-space** \(\mathbb{P}^n\) is the set of lines through the origin in \(\mathbb{C}^{n+1}\).

The *homogeneous “coordinate”* of a point in \(\mathbb{P}^n\) (\(=\) line in \(\mathbb{C}^{n+1}\)) is:

\[(x_1 : \cdots : x_{n+1})\text{ (not all zero)}\]

and it is well-defined modulo:

\[(x_1 : \cdots : x_{n+1}) = (\lambda x_1 : \cdots : \lambda x_{n+1}) \text{ for } \lambda \in \mathbb{C}^*\]

Projective \(n\)-space is an overlapping union:

\[\mathbb{P}^n = \bigcup_{i=1}^{n+1} U_i; \quad U_i = \{(x_1 : \cdots : x_m : \cdots x_{n+1}) | x_m \neq 0\} = \mathbb{C}^n\]

and a disjoint union:

\[\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \mathbb{C}^{n-2} \cup \cdots \cup \mathbb{C}^0\]

where \(\mathbb{C}^{n-m} = \{(x_1 : \cdots : x_{n+1}) | x_1 = \cdots = x_m = 0, x_{m+1} \neq 0\}\)

The polynomial ring is graded by degree:

\[\mathbb{C}[x_1, \ldots, x_{n+1}] = \bigoplus_{d=0}^{\infty} \mathbb{C}[x_1, \ldots, x_{n+1}]_d\]

and the nonzero polynomials \(F \in \mathbb{C}[x_1, \ldots, x_{n+1}]_d\) are the *homogeneous polynomials of degree \(d\)*. The value of a homogeneous polynomial \(F\) of degree \(d\) at a point \(x \in \mathbb{P}^n\) is not well-defined, since:

\[F(\lambda x_1, \ldots, \lambda x_{n+1}) = \lambda^d F(x_1, \ldots, x_{n+1})\]

However, the *locus of zeroes* of \(F\) is well-defined, hence \(F\) determines a *projective hypersurface* if \(d > 0\):

\[V(F) = \{(x_1 : \cdots : x_{n+1}) \in \mathbb{P}^n | F(x_1 : \cdots : x_{n+1}) = 0\} \subset \mathbb{P}^n,\]

**Definition:** An ideal \(I \subset \mathbb{C}[x_1, \ldots, x_{n+1}]\) is *homogeneous* if:

\[I = \bigoplus I_d; \text{ where } I_d = I \cap \mathbb{C}[x_1, \ldots, x_{n+1}]_d\]

equivalently, \(I\) has (finitely many) homogeneous generators.
Corollary 3.1: For a homogeneous ideal $I \subset \mathbb{C}[x_1, \ldots, x_{n+1}]$,

\[ V(I) = \{(x_1 : \cdots : x_{n+1}) \mid F(x_1 : \cdots : x_{n+1}) = 0 \text{ for all } F \in I\} \subset \mathbb{P}^n \]

is an intersection of finitely many projective hypersurfaces.

The sets $V(I) \subset \mathbb{P}^n$ are the algebraic subsets of $\mathbb{P}^n$. The irreducible algebraic sets are defined as in the case of affine varieties, and satisfy:

\[ X = V(\mathcal{P}) \subset \mathbb{P}^n \]

for a unique homogeneous prime ideal $\mathcal{P} \subset \mathbb{C}[x_1, \ldots, x_{n+1}]$. Thus each irreducible algebraic set inside $\mathbb{P}^n$ has a homogeneous coordinate ring:

\[ R(X) := \mathbb{C}[x_1, \ldots, x_n]/\mathcal{P} = \bigoplus \mathbb{C}[x_1, \ldots, x_n]/\mathcal{P}_d \]

**Warning:** Unlike the coordinate rings of isomorphic affine varieties, homogeneous rings of isomorphic projective varieties will not usually be isomorphic graded rings. Even more fundamentally, a non-constant element of the homogeneous ring of $X \subset \mathbb{P}^n$ is not a function.

(In fact, homogeneous rings are made up of sections of line bundles.)

**Note:** There is one homogeneous maximal ideal, namely:

\[ \langle x_1, \ldots, x_{n+1} \rangle \subset \mathbb{C}[x_1, \ldots, x_{n+1}] \]

This is usually called the irrelevant homogeneous ideal, and it contains all other homogeneous ideals.

**Definition:** A homogeneous ideal $m \subset \mathbb{C}[x_1, \ldots, x_{n+1}]$ is homaximal if it is maximal among all homogeneous ideals other than $\langle x_1, \ldots, x_{n+1} \rangle$.

**Exercises 3.1:**

(a) Given a homogeneous $I \subset \langle x_1, \ldots, x_{n+1} \rangle$, there is a bijection:

\{homaximal ideals $m_x$ containing $I$\} $\leftrightarrow$ \{points $x \in V(I) \subset \mathbb{P}^n$\}

(b) There is a (Zariski) topology on $X = V(\mathcal{P})$ generated by:

\[ U_F := X - V(F) \subset \mathbb{P}^n \]

for $F \in R(X)_d$ consisting of open sets of the form $U_I := X - V(I)$ for homogeneous ideals $I \subset R(X)$.

**Definition:** The field of rational functions on $X = V(\mathcal{P}) \subset \mathbb{P}^n$ is:

\[ \mathbb{C}(X) := \left\{ \frac{F}{G} \mid F, G \in R(X)_d \text{ for some } d, \text{ and } G \neq 0 \right\} / \sim \]

**Good News:** Rational functions are $\mathbb{C}$-valued functions on some $U$.

**Bad News/Exercise 3.2:** The only rational functions defined everywhere on $X$ are the constant functions.
Proposition 3.2: The sheaf $\mathcal{O}_X$ of $\mathbb{C}$-valued functions on (the open sets of) an irreducible algebraic set $X = V(\mathcal{P}) \subset \mathbb{P}^n$ defined by:

$$\mathcal{O}_x := \left\{ \frac{F}{G} \mid G(x) \neq 0 \right\} \subset \mathbb{C}(X)$$

and $\mathcal{O}_X(U) := \bigcap_{x \in U} \mathcal{O}_x \subset \mathbb{C}(X)$ gives $(X, \mathcal{O}_X)$ the structure of a prevariety.

Proof: We need to prove $(X, \mathcal{O}_X)$ is locally affine.

For each $i = 1, \ldots, n+1$, either:

(a) $x_i \in \mathcal{P}$, in which case $X \cap U_i = \emptyset$, or else

(b) $x_i \not\in \mathcal{P}$, in which case $U_{x_i} = X \cap U_i$, with $\mathcal{O}_X|_{U_{x_i}}$ is isomorphic to the affine variety corresponding to the $\mathbb{C}$-algebra:

$$\mathbb{C}[U_{x_i}] = \mathbb{C}\left[\frac{x_1}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right]/\overline{\mathcal{P}},$$

where $F(x_1/x_i, \ldots, x_{n+1}/x_i) \in \overline{\mathcal{P}} \iff F \in \mathcal{P}$

which, incidentally, satisfies $\mathbb{C}(U_{x_i}) = \mathbb{C}(X)$. Since $X$ is covered by these open sets, it follows that $X$ is locally affine.

Definition: A prevariety is projective if it is isomorphic to one of the $X = V(\mathcal{P}) \subset \mathbb{P}^n$ with sheaf of $\mathbb{C}$-valued functions defined as above.

As for separatedness, first notice:

Proposition 3.3: $\mathbb{P}^m \times \mathbb{P}^n$ is projective, and $\mathbb{P}^n$ is separated.

Proof: The Segre embedding is the map (of sets):

$$\sigma : \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^{(n+1)(m+1)-1}$$

$$((x_1 : \cdots : x_{n+1}), (y_1 : \cdots : y_{m+1})) \mapsto (\cdots : x_i y_j : \cdots)$$

(We will use $z_{ij}$ as homogeneous coordinates for points of $\mathbb{P}^{(n+1)(m+1)-1}$.)

The image of $\sigma$ is the irreducible algebraic set:

$$X_{m,n} := V(\{z_{ij}z_{kl} - z_{il}z_{kj}\}) \subset \mathbb{P}^{(n+1)(m+1)-1}$$

and moreover, as a prevariety, $X_{m,n}$ (with sheaf of regular functions) is the product of $\mathbb{P}^n$ and $\mathbb{P}^m$. Also, if $n = m$, then:

$$\delta(\mathbb{P}^n) = X_{n,n} \cap V(\{z_{ij} - z_{ji}\})$$

is closed, so $\mathbb{P}^n$ is separated.

Exercise 3.3: (a) Carefully show that the projection $\pi_{\mathbb{P}^m} : X_{m,n} \rightarrow \mathbb{P}^m$ is a morphism of prevarieties.

(b) Extend Proposition 3.3 to describe the product of projective prevarieties $X = V(\mathcal{P}) \subset \mathbb{P}^n$ and $Y = V(\mathcal{Q}) \subset \mathbb{P}^m$ as an irreducible, closed subset of $X_{m,n}$, hence it is a projective prevariety, and then conclude that all projective prevarieties are varieties.
Definition: An open subset $U \subset X$ of a projective variety, together with the induced sheaf $\mathcal{O}_X|_U$ of $\mathbb{C}$-valued functions, or more generally, any variety isomorphic to such a pair $(U, \mathcal{O}_X|_U)$, is quasi-projective.

Proposition 3.4: A quasi-affine variety is also quasi-projective.

Proof: It suffices to show that each affine variety is quasi-projective. To this end, suppose $(Y, \mathcal{O}_Y)$ is isomorphic to the affine variety obtained from $X = V(\mathcal{P}) \subset \mathbb{C}^n$. Then we may identify $\mathbb{C}^n$ with $U_{n+1} \subset \mathbb{P}^n$ and take the (Zariski) closure $\overline{X}$ of $X \subset \mathbb{P}^n$. Then

$$\overline{X} = V(\mathcal{P}^h), \text{ where } \mathcal{P}^h = \langle f^h \mid f \in \mathcal{P} \rangle$$

and $f^h(x_1, \ldots, x_{n+1}) := x_{n+1}^d f\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right)$ whenever $d = \deg(f)$. Check that $\mathcal{P}^h$ defined this way is a prime ideal, and that $\overline{X} \cap \mathbb{C}^n = X$.

Exercise 3.4: Find generators $\langle f_1, \cdots, f_m \rangle = \mathcal{P}$ of a prime ideal with the property that the $f_i^h$ do not generate $\mathcal{P}^h$.

Definition: In a category whose objects are topological spaces, whose morphisms are continuous, and in which products exist, a separated object $X$ is proper if “projecting from $X$ is universally closed,” i.e.

$$\pi_Y : X \times Y \to Y$$

maps closed sets $Z \subset X \times Y$ to closed sets $\pi_Y(Z) \subset Y$, for all $Y$.

Exercise 3.5: In the category of topological spaces, compact implies proper, and conversely, any proper space with the property that every open cover has a countable subcover is also compact.

Concrete Example: Suppose $(X, \mathcal{O}_X)$ is affine and $f \in \mathbb{C}[X]$ is a non-constant function. Then the hyperbola over $U_f$:

$$V(xf - 1) \subset X \times \mathbb{C}$$

is closed, but its projection to $\mathbb{C}$ has image $\mathbb{C}^*$, which is not closed. Thus, the only proper affine variety is the one-point space.

General Example: If $U \subset X$ is an open subset of a separated object and $U \neq U$ (e.g. any quasi-projective variety properly contained in a projective variety), then $U$ is not proper. Indeed, the closed diagonal in $X \times X$ determines a closed set:

$$\Delta \cap (U \times X)$$

that projects to $U \subset X$.

Theorem (Grothendieck) Projective varieties are proper varieties.

To prove this, we need a result from commutative algebra:
Nakayama’s Lemma (Version 1): Suppose \( M \) is a finitely generated module over a ring \( A \) and \( I \subset A \) is an ideal such that:

\[
IM = M
\]

Then there an element \( a \in I \) such that \((1 + a)m = 0\) for all \( m \in M \).

**Proof:** Let \( m_1, \ldots, m_n \in M \) be generators. Then \( IM = M \) gives:

\[
m_i = \sum_{j=1}^{n} a_{ij} m_j \quad \text{for} \quad a_{ij} \in I
\]

implying that the matrix \( 1 - (a_{ij}) \) has a kernel, hence \( b = \det(1 - (a_{ij})) \) satisfies \( bm_i = 0 \) for all \( i \), and evidently, \( b = 1 + a \) for some \( a \in I \).

**Proof:** (of the theorem) It suffices to prove that for all \( n, m \):

\[
\pi_{C^m} : \mathbb{P}^n \times \mathbb{C}^m \rightarrow \mathbb{C}^m \quad \text{is a closed map}
\]

Let \( \mathbb{C}[x_1, \ldots, x_{n+1}] = R(\mathbb{P}^n) \) and \( \mathbb{C}[y_1, \ldots, y_m] = \mathbb{C}[\mathbb{C}^m] \), and suppose \( Z \subset \mathbb{P}^n \times \mathbb{C}^m \) is closed

Then we need to show that \( \pi_{C^m}(Z) = V(J) \) for some \( J \subset \mathbb{C}[y_1, \ldots, y_m] \).

To this end, grade the ring \( \mathbb{C}[\overline{x}, \overline{y}] := \mathbb{C}[x_1, \ldots, x_{n+1}, y_1, \ldots, y_m] \) by degree in the \( x \)-variables:

\[
\mathbb{C}[\overline{x}, \overline{y}] = \bigoplus_d \mathbb{C}[x_1, \ldots, x_{n+1}]_d \otimes \mathbb{C}[y_1, \ldots, y_m]
\]

and consider the subsets defined by homogeneous ideals:

\[
V(I) \subset \mathbb{P}^n \times \mathbb{C}^m; \quad I = \bigoplus_d I_d \subset \bigoplus_d \mathbb{C}[\overline{x}, \overline{y}]_d
\]

Then the theorem follows immediately from:

**Claim 1:** Every closed set \( Z \subset \mathbb{P}^n \times \mathbb{C}^m \) is equal to some \( V(I) \).

**Claim 2:** \( \pi_{C^m}(V(I)) = V(I_0) \), the “degree zero” part of the ideal \( I \).

**Proof of Claim 1:** Cover \( \mathbb{P}^n \) by the open sets \( U_i = \mathbb{C}^n \times \mathbb{C}^m \) with:

\[
\mathbb{C}[U_i] = \mathbb{C}[\frac{x_1}{x_i}, \ldots, \frac{x_{n+1}}{x_i}, y_1, \ldots, y_m]
\]

and, given a closed set \( Z \subset \mathbb{P}^n \times \mathbb{C}^m \), define a homogeneous ideal \( I \) by:

\[
I_d := \left\{ F \in \mathbb{C}[\overline{x}, \overline{y}]_d \mid \frac{F}{x_i^d} \in I(Z \cap U_i) \subset \mathbb{C}[U_i] \text{ for all } i \right\}
\]

It is clear that \( Z \subset V(I) \). For the other inclusion, suppose \( (a, b) \notin Z \). Then \( (a, b) \in U_i \) for some \( i \), so there is an \( f \in I(Z \cap U_i) \) such that \( f(a, b) \neq 0 \). It follows that \( x_i^d f \in \mathbb{C}[U_i] \) for some \( d \), and then that \( F := x_i^{d+1} f \in I_d \). Since \( F(a, b) \neq 0 \), it follows that \( (a, b) \notin V(I) \).
Proof of Claim 2: Again, the inclusion $\pi_{C^m}(Z) \subset V(I_0)$ is clear. For the other inclusion, suppose $b = (b_1, \ldots, b_m) \not\in \pi_{C^m}(Z)$, and let $m_b = (y_1 - b_1, \ldots, y_m - b_m)$ be the corresponding maximal ideal. Then:

\[ Z \cap (\mathbb{P}^n \times b) = \emptyset, \text{ so } (Z \cap U_i) \cap (U_i \times b) = \emptyset \text{ for all } i \]

which in turn implies that:

\[ I(Z \cap U_i) + \mathbb{C} \left[ \frac{x_1}{x_i}, \ldots, \frac{x_{n+1}}{x_i} \right] \cdot m_b = \mathbb{C}[U_i] \text{ for all } i \]

and thus for each $i = 1, \ldots, n+1$, there exist $f_i \in I(Z \cap U_i), g_{ij} \in \mathbb{C}[U_i]$ and $m_{ij} \in m_b$ such that $f_i + \sum g_{ij} m_{ij} = 1$. Moreover, by multiplying through by a sufficiently large power $d_i$ of each $x_i$, we can arrange that:

\[ F_i + \sum_j G_{ij} m_{ij} = x_i^{d_i} \quad \text{for } F_i \in I_{d_i}, G_{ij} \in \mathbb{C}[[x, y]]_{d_i} \]

If we moreover take $d > \sum d_i$, then we have $I_d + \mathbb{C}[[x, y]]_{d} \cdot m_b = \mathbb{C}[[x, y]]_{d}$.

Thus the finitely generated $\mathbb{C}[y_1, \ldots, y_m]$ -modules:

\[ M_d := \mathbb{C}[[x, y]]/I_d \text{ satisfy } m_b \cdot M_d = M_d \]

hence by Nakayama’s lemma, there is an $f \in \mathbb{C}[y_1, \ldots, y_m]$ such that $f(b) \neq 0$ and $f M_d = 0$, i.e. $f \cdot \mathbb{C}[[x, y]]_{d} \in I_d$. But this implies $f \cdot x_i^{d_i} \in I_d$ for all $i$, from which it follows that $f \in I_0$, as desired.

Corollary 3.5/Exercise 3.6: Any morphism $\Phi : X \to Y$ from a projective variety $X$ to an abstract variety $Y$ is a closed mapping.

4. Toric Varieties. A toric variety is an abstract variety obtained by gluing affine varieties that are defined by “combinatorial data.” Although these varieties all satisfy $\mathbb{C}(X) = \mathbb{C}(x_1, \ldots, x_n)$, and in fact all contain the affine “torus” associated to $\mathbb{C}(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})$ as a common open set, toric varieties are a rich source of examples.

For Motivation: Consider again the basic open sets in $\mathbb{P}^2$, satisfying:

\[ \mathbb{C}[U_1] = \mathbb{C} \left[ 1, \frac{x_2}{x_1}, \frac{x_3}{x_1} \right], \quad \mathbb{C}[U_2] = \mathbb{C} \left[ \frac{x_1}{x_2}, 1, \frac{x_3}{x_2} \right] \quad \text{and} \quad \mathbb{C}[U_3] = \mathbb{C} \left[ \frac{x_1}{x_3}, \frac{x_2}{x_3}, 1 \right] \]

and glued along their intersections:

\[ \mathbb{C}[U_1 \cap U_2] = \mathbb{C} \left[ \frac{x_2}{x_1}, \frac{x_1}{x_2}, \frac{x_3}{x_1}, \frac{x_3}{x_2} \right], \text{ etc.} \]

and triple intersection satisfying:

\[ \mathbb{C}[U_1 \cap U_2 \cap U_3] = \mathbb{C} \left[ \frac{x_1}{x_2}, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_1}{x_3}, \frac{x_3}{x_2}, \frac{x_2}{x_3} \right] \]
Notice that some of the generators can be dropped. In fact, define:

\[ x := \frac{x_2}{x_1} \quad \text{and} \quad y := \frac{x_3}{x_1} \]

and redo the rings above in terms of these new variables:

\[ C[U_1] = \mathbb{C}[x, y], \quad C[U_2] = \mathbb{C}[x^{-1}, yx^{-1}], \quad C[U_3] = \mathbb{C}[y^{-1}, xy^{-1}] \]

\[ C[U_1 \cap U_2] = \mathbb{C}[x, x^{-1}, y], \quad C[U_1 \cap U_3] = \mathbb{C}[x, y, y^{-1}] \]

\[ C[U_2 \cap U_3] = \mathbb{C}[x^{-1}, y^{-1}, x^{-1}y, xy^{-1}] \]

and

\[ C[U_1 \cap U_2 \cap U_3] = \mathbb{C}[x, x^{-1}, y, y^{-1}] \]

If we identify the integer lattice in the (real) plane with monomials:

\[ (m, n) \leftrightarrow x^m y^n \]

then the monomials in the various rings are the integer points of cones. When we pass to the dual cones, however, we get a neat picture of the gluing, with the open sets \( U_i \) represented by cones with union \( \mathbb{R}^2 \), and the intersections represented by intersections of the cones.