

Polynomials.

Math 4800/6080 Project Course

1. Introduction. Polynomials in one variable are already a rich source of mathematics. Why study polynomials in more variables? Here are three of many possible reasons.

- (1) Polynomials are functions. One variable polynomials only have a one-dimensional domain, but our world is three-dimensional, and our minds can imagine any number of dimensions. As with functions of one variable, a “reasonable” function can be well approximated by a Taylor polynomial, so polynomials in several variables can be used to model more interesting phenomena.
- (2) The level sets of a polynomial in two variables are curves, and in three variables they are surfaces. These are shapes with very interesting geometric (and “arithmetic”) properties.
- (3) Polynomials can provide useful information. For example, the determinant of an $n \times n$ matrix is a polynomial in the entries of the matrix. A matrix is a *transformation* of n -space, and by Cramer’s rule, this transformation has an inverse if and only if its determinant is not zero.

The **conic** is certainly the most familiar polynomial level set:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

where x, y are the variables, and a, b, c, d, e, f are the coefficients. We will review these later, and “close them up” in the projective plane. We will contrast them with the less familiar (but important) plane cubics:

$$ax^3 + bx^2y + cxy^2 + dy^3 + ex^2z + fxz^2 + gz^3 + hy^2z + hz^2 + jxyz = 0$$

among which are the *elliptic curves* (of Fermat’s Last Theorem fame).

One celebrated example of a conic equation is **Pell’s Equation**

$$x^2 - dy^2 = 1$$

where d is a non-square positive integer. The set of real solutions to this is a hyperbola, the set of rational solutions is a dense subset of the hyperbola, but the integer solutions are perhaps the most interesting (and fascinated number theorists of the 17th and 18th centuries).

It turns out that there are infinitely many integer solutions. Finding the **smallest** integer solution other than the two obvious ones $(\pm 1, 0)$ can be quite a challenging problem.

When $d = 2$, it isn't so bad. By inspection, $(3, 2)$ is a solution, since:

$$3^2 - 2(2^2) = 9 - 8 = 1$$

Once we have this solution, we can create infinitely many with a trick. Rewrite the equation above as:

$$3^2 - 2(2^2) = (3 + 2\sqrt{2})(3 - 2\sqrt{2}) = 1$$

and then square both sides to get:

$$1^2 = (3 + 2\sqrt{2})^2(3 - 2\sqrt{2})^2 = (17 + 12\sqrt{2})(17 - 12\sqrt{2}) = (17)^2 - 2(12)^2$$

since $(3 \pm 2\sqrt{2})^2 = 9 \pm 12\sqrt{2} + 8 = 17 \pm 12\sqrt{2}$.

Thus we get another solution $(17, 12)$. But why stop with squaring? We can cube both sides, take fourth powers, etc. to get infinitely many solutions (and up to signs, these turn out to be all of them).

Polynomials are linear combinations of finitely many **monomials**. The monomials in one variable are just the powers of the variable:

$$x^d$$

In two variables, the monomials are:

$$x^i y^j$$

and in three variables:

$$x^i y^j z^k$$

while in n variables, we'll need to use subscripts:

$$(*) \quad x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

The **degree** of the monomial $(*)$ is:

$$i_1 + i_2 + \cdots + i_n$$

so that, looking back, we see that the conic polynomial:

$$ax^2 + bxy + cy^2 + dx + ey + f$$

is a linear combination of monomials of degrees 2, 1 and 0 while the cubic is a linear combination of monomials of degrees 3, 2, 1 and 0.

Here is an important

Definition. A polynomial is *homogeneous of degree d* if it is a linear combination of monomials of degree d .

Unlike the space of all polynomials, the space of all homogeneous polynomials of a given degree d in n variables is *finite dimensional*. How many dimensions does it have? Let's try some examples.

One Variable. The homogeneous polynomials of degree d in one variable are scalar multiples of:

$$x^d$$

This is one-dimensional, for any degree.

Two Variables. The homogeneous polynomials of degree d in two variables are linear combinations of:

$$x^d, x^{d-1}y, x^{d-2}y^2, \dots, xy^{d-1}, y^d$$

There are $d+1$ of them, and they are a basis, so the dimension is $d+1$.

Three Variables. In degree zero, there is one monomial, namely 1. The next few degrees yield:

(a) Degree one. The three monomials of degree one are:

$$x, y \text{ and } z$$

(b) Degree two. There are 6 monomials of degree two:

$$x^2, xy, y^2, xz, yz, \text{ and } z^2$$

(c) Degree three. There are 10 monomials of degree three:

$$x^3, x^2y, xy^2, y^3, x^2z, xyz, y^2z, xz^2, yz^2 \text{ and } z^3$$

Recall that 1, 3, 6, 10 are the first four *triangle numbers*.

Proposition. The number of monomials of degree d in 3 variables is:

$$\binom{d+2}{2}$$

Proof. Group the monomials of degree d by the power of z :

z^d (there is one of these)

xz^{d-1}, yz^{d-1} (two of these)

$x^2z^{d-2}, xyz^{d-2}, y^2z^{d-2}$ (three of these)

$\vdots$

$x^iz^{d-i}, x^{i-1}yz^{d-i}, \dots, xy^{i-1}z^{d-i}, y^iz^{d-i}$ ($i+1$ of these)

$\vdots$

$x^dz^0, x^{d-1}yz^0, \dots, xy^{d-1}, y^dz^0$ ($d+1$ of these).

so all together, in three variables and degree d , there are:

$$1 + 2 + 3 + \dots + (d+1) = \binom{d+2}{2} \text{ monomials}$$

Adding symmetry to the definition yields a more subtle question.

Definition. A (homogeneous) polynomial P is **symmetric** if it is unchanged when the variables are permuted.

Examples. (a) A *monomial* is symmetric if every power is the same:

$$x^i y^i \text{ or } x^i y^i z^i \text{ or, more generally, } x_1^i x_2^i \cdots x_n^i$$

are the symmetric monomials.

(b) In two variables, the symmetric homogeneous polynomials are the linear combinations of:

1 in degree zero,

$x + y$ in degree one

$x^2 + y^2$ and xy in degree two

$x^3 + y^3$ and $x^2y + xy^2$ in degree three

$x^4 + y^4$, $x^3y + xy^3$ and x^2y^2 in degree four, etc.

What would be a good basis (analogous to the monomials) for the symmetric homogeneous polynomials of degree d in 3 or more variables? We will explore this and related questions.

Exercises. 1. Using induction (on the number of variables) and:

$$\binom{d+k}{k} = \binom{k-1}{k-1} + \binom{k}{k-1} + \cdots + \binom{d+k-1}{k-1}$$

conclude that the number of monomials of degree d in n variables is:

$$\binom{d+n-1}{n-1}$$

2. Minor changes to Pell's equation have major impacts. Show that:

(a) The hyperbola of real solutions to:

$$x^2 - 4y^2 = 1$$

contains a dense set of rational solutions but only two integer solutions.

(b) The hyperbola of real solutions to:

$$x^2 - 2y^2 = 3$$

contains **no** integer solutions, and **no** rational solutions!

(c) The circle of real solutions to:

$$x^2 + y^2 = d$$

contains either a dense set of rational solutions, or no rational solutions, depending upon d . It never has an infinite set of integer solutions.