Polynomials.
Math 4800/6080 Project Course

2. Algebraic Curves.

"Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions."

–Felix Klein

A plane curve is the level (zero) set:

$$(p = 0) = \{(x, y) \in \mathbb{R}^2 \mid p(x, y) = 0\} \subset \mathbb{R}^2$$

of a polynomial in two variables, *with some exceptions*.

A projective plane curve is the zero set:

$$(P = 0) = \{(x : y : z) \in \mathbb{RP}^2 \mid P(x : y : z) = 0\} \subset \mathbb{RP}^2$$

of a *homogeneous* polynomial in three variables *with some exceptions*.

Remark. Recall that the coordinate of a point in $\mathbb{RP}^2$ is a ratio:

$$(x : y : z)$$

A monomial $x^i y^j z^k$ of degree d in x, y, z satisfies:

$$(\alpha x)^i (\alpha y)^j (\alpha z)^k = \alpha^d (x^i y^j z^k)$$

and therefore a homogeneous polynomial $P(x, y, z)$ of degree d satisfies:

$$P(\alpha x, \alpha y, \alpha z) = \alpha^d P(x, y, z)$$

and in particular, $P(\alpha x, \alpha y, \alpha z) = 0 \Leftrightarrow P(x, y, z) = 0$.

Therefore it is **well-defined** to write $P(x : y : z) = 0$ even though it is not well-defined to write $P(x : y : z) = c$ for any other constant.

The Completion of a Plane Curve. Given a polynomial $p(x, y)$ made up of monomials of degree d or less, the *homogenization* of $p(x, y)$ is the homogeneous polynomial $P(x, y, z)$ in three variables defined by inserting z's to the necessary power to raise the degree of every monomial to exactly d. Precisely, this is achieved by setting:

$$P(x, y, z) := z^d \cdot p(x/z, y/z)$$

Thinking of $\mathbb{RP}^2$ as the real plane with points at infinity:

$$(*) \quad \mathbb{RP}^2 = \mathbb{R}^2 \cup \mathbb{RP}^1; \quad \mathbb{R}^2 = \{(x : y : 1)\}, \quad \mathbb{RP}^1 = \{(x : y : 0)\}$$

$$(P = 0) \cap \mathbb{R}^2 = (P(x, y, 1) = 0) = (p(x, y) = 0) \text{ and}$$

$$(P = 0) \cap \mathbb{RP}^1 = (P(x : y : 0) = 0) \text{ is a finite set of points}$$

In this sense, $(P = 0)$ should be thought of as the *completion* of $(p = 0)$ by adding the extra points $(P(x : y : 0) = 0)$ at infinity.

Examples. (a) The linear (non-homogeneous) polynomial:

$$p(x, y) = ax + by + c$$

homogenizes to $P(x, y, z) = ax + by + cz$ with one point at infinity:

$$(P(x : y : 0 = 0) = \{(x : y : 0) \mid ax + by = 0\} = \{(b : -a)\}$$

(b) The quadratic polynomial defining the hyperbola:

$$p(x, y) = xy - 1$$

homogenizes to $P(x, y, z) = xy - z^2$ with two points to add at infinity:

$$(P(x : y : 0) = 0) = \{(x : y : 0) \mid xy = 0\} = \{(1 : 0 : 0), (0 : 1 : 0)\}$$

Notice that with these two points added, $(P = 0)$ becomes a circle.

(c) The quadratic polynomial defining the unit circle:

$$p(x, y) = x^2 + y^2 - 1$$

homogenizes to $P(x, y, z) = x^2 + y^2 - z^2$ with no added points at infinity:

$$(P(x : y : 0) = 0) = \{(x : y : 0) \mid x^2 + y^2 = 0\} = \emptyset$$

Curve Exceptions. Here are two of the exceptions Klein speaks of:

(i) A (projective) plane curve should not have any isolated points.

(ii) A (projective) plane curve should not be empty.

Examples. (i) The polynomial $p(x, y) = x^2 + y^2 - x^3$ does not define a real plane curve because it has an isolated point:

$$(p = 0) = \{(0, 0)\} \cup \{\text{a curve of points } (x, y) \text{ with } x \geq 1\}$$

Passing to the homogeneous polynomial:

$$(x^2 z + y^2 z - x^3 = 0) = (p = 0) \cup \{(0 : 1 : 0)\}$$

adds a point at infinity to "complete" the unbounded component of the plane curve, but the point $\{(0, 0)\}$ remains isolated.

(ii) The polynomial: $p(x, y) = x^2 + y^2 + 1$ vanishes nowhere. The projective completion is also empty:

$$(x^2 + y^2 + z^2 = 0) = \emptyset$$

There will be some more subtle exceptions later.

Here is an interesting class of higher degree examples of plane curves:

Hyperelliptic Curves. Polynomials of degree $d \geq 3$ of the form:

$$p(x,y) = y^2 - (x-r_1)(x-r_2)\cdots(x-r_d) \text{ with } r_1 < r_2 < \cdots < r_d$$

come in two flavors:

(a) (odd values of d) The plane curve $(p=0)$ consists of $(d-1)/2$ "ovals" with pairs of x-intercepts:

$$(r_{2i-1}, 0) \text{ and } (r_{2i}, 0)$$

followed by an unbounded component with $(r_d, 0)$ as its only x-intercept.

(b) (even values of d) The plane curve $(p=0)$ has left and right unbounded components with x-intercepts $(r_1, 0)$ and $(r_d, 0)$, respectively, as well as $(d-2)/2$ intermediate ovals, each with two x-intercepts.

When we homogenize these polynomials, we get:

$$P = y^2 z^{d-2} - (x - zr_1)\cdots(x - zr_d)$$

with a single point at infinity:

$$(P(x:y:0) = 0) = \{(x:y:0) \mid -x^d = 0\} = \{(0:1:0)\}$$

because we are assuming that $d \geq 3$. (What happens when $d = 2$?)

Evidently, a (projective) plane curve usually looks locally like a bit of the real line. There may be crossing points and "cusps" where the plane curve looks a bit different. These are *singularities* of a curve:

Singularities. Let $C = (p = 0)$ and write out the polynomial p:

$$p(x,y) = p_e(x,y) + \cdots + p_d(x,y)$$

as a sum of homogeneous polynomials of degrees e through d.

Definition. (a) The *multiplicity* of C at $(0,0)$ is e.

(Note that $(0,0) \in C$ if and only if $e \geq 1$ (i.e. p has no constant term)).

(b) C is *nonsingular* at $(0,0)$ if $e = 1$, in which case

$$(p_1(x,y) = 0) \subset \mathbb{R}^2$$

is the *tangent line*.

(c) $(0,0)$ is a *singular point* of C if $e > 1$.

Examples. The curves defined by:

$$p(x,y) = y^2 - x^2 - x^3 \quad \text{and} \quad q(x,y) = y^2 - x^3$$

are both singular at $(0,0)$. The first example is a crossing point, and the second is a cusp. This can be detected by examining their first homogeneous terms: $y^2 - x^2$ and y^2, which are, in the first case, a pair of intersecting lines, and in the second, the x-axis "counted twice."

Definition. A singularity of $C = (p = 0)$ at $(0, 0)$ is *real* if $p_e(x, y)$ factors as a product of linear homogeneous polynomials, in which case:

$$(p_e(x, y) = 0)$$

(counting lines with multiplicity) is the *tangent cone* of C at $(0, 0)$.

A Subtle Klein Exception. If C has a singularity that is not real, then it shouldn't be counted as a plane curve.

Examples. Consider the polynomials:

$$p(x, y) = x(x^2 + y^2) - y^4 \text{ and } p(x, y) = x(x^2 - y^2) - y^4$$

In the first case, $p_3(x, y) = x(x^2 + y^2)$ does not factor over the reals, but in the second, $p_3(x, y) = x(x^2 - y^2) = x(x - y)(x + y)$ **does** factor. So the second would be considered a legitimate real plane curve.

Singularities away from the origin. If $(x_0, y_0) \in \mathbb{R}^2$ is a point other than the origin, then we may *translate* that point to the origin and then compute the singularity. However, there is another way to use calculus to do this directly.

Definition. If:

(0) $p(x_0, y_0) \neq 0$, the multiplicity of $C = (p = 0)$ at (x_0, y_0) is zero.

(1) $p(x_0, y_0) = 0$ but either of the first partials:

$$\frac{\partial p}{\partial x}(x_0, y_0) \text{ or } \frac{\partial p}{\partial y}(x_0, y_0) \text{ (or both)}$$

are not zero, then the multiplicity of C at (x_0, y_0) is one, and:

$$(\frac{\partial p}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial p}{\partial y}(x_0, y_0) \cdot (y - y_0) = 0)$$

is the tangent line in $\mathbb{R}^2$ to C at the point (x_0, y_0).

(e) If all the mixed partials of $p(x, y)$ of degree $e - 1$ or less vanish at (x_0, y_0), but **some** mixed partial of degree e fails to vanish at (x_0, y_0), then the multiplicity of C at (x_0, y_0) is e.

Observation. If $(x_0, y_0) = (0, 0)$, then the two definitions agree.

Singularities at Infinity. The points at infinity may be analyzed for singularites. To do this, we may either move them to the finite region via one of the permutation projective transformations, or else define singularities directly at **all** points of the projective plane in terms of the homogeneous polynomial $P(x, y, z)$. We will take the second approach.

Euler's Lemma. If $P(x, y, z)$ is homogeneous of degree d, then

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z} = d \cdot P$$

Proof. Euler's Lemma is clearly true for *monomials*.

If $P = x^i y^j z^k$, then:

$$\frac{\partial P}{\partial x} = ix^{i-1}y^j z^k, \frac{\partial P}{\partial y} = jx^i y^{j-1} z^k \text{ and } \frac{\partial P}{\partial z} = kx^i y^j z^{k-1}$$

and so:

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z} = i(x^i y^j z^k) + j(x^i y^j z^k) + k(x^i y^j z^k) = d \cdot P$$

But a homogeneous polynomial of degree d is a linear combination of monomials, and both the left and right side of Euler's Lemma hold for a linear combinations of monomials once they hold for monomials.

Corollary. If all the partials of the *homogeneous polynomial* P vanish:

$$\frac{\partial P}{\partial x}(x_0 : y_0 : z_0) = \frac{\partial P}{\partial y}(x_0 : y_0 : z_0) = \frac{\partial P}{\partial z}(x_0 : y_0 : z_0) = 0$$

then $(x_0 : y_0 : z_0) \in (P = 0)$.

On the other hand, if $(x_0 : y_0 : z_0) \in (P = 0)$ and at least one of the partial derivatives is non-zero, then:

$$(x\frac{\partial P}{\partial x}(x_0 : y_0 : z_0) + y\frac{\partial P}{\partial y}(x_0 : y_0 : z_0) + z\frac{\partial P}{\partial x}(x_0 : y_0 : z_0) = 0)$$

passes through the point $(x_0 : y_0 : z_0)$ and is the tangent line to the projective curve at that point.

Definition. $(x_0 : y_0 : z_0)$ is a singular point of $(P = 0)$ if

$$\frac{\partial P}{\partial x}(x_0 : y_0 : z_0) = \frac{\partial P}{\partial y}(x_0 : y_0 : z_0) = \frac{\partial P}{\partial z}(x_0 : y_0 : z_0) = 0$$

More generally, $(x_0 : y_0 : z_0)$ is a point of multiplicity e on $(P = 0)$ if all mixed partials up to order $e - 1$ vanish at the point, but some mixed partial of order e fails to vanish.

Remark: It is a little bit involved, but this definition matches the definition of multiplicity for points of $(p = 0)$.