Categories, Symmetries and Manifolds

Math 4800, Fall 2020

0. Introduction. **Symmetry** in mathematics has a context, captured by the notion of a **category**. In this course we will revisit the undergraduate mathematics curriculum in algebra and geometry through the lenses of symmetry and categories. This will culminate in the classification of the representations of finite groups via their *character tables*. We will also study categories of **differentiable manifolds**, a graduate topic building on calculus and analysis that can used, among other things, to help classify representations of complex groups arising in physics.

A category axiomatizes some basic properties of sets and functions between sets. If S, T and U are sets and

$$f : S \to T \text{ and } g : T \to U$$

are functions, then $g \circ f : S \to U$ is their **composition**, defined by:

$$(g \circ f)(s) = g(f(s)) \text{ for all } s \in S$$

Composition is thus an *operation* on pairs of functions that is rarely commutative; unless U and S are the same set, the reverse composition $f \circ g$ isn't even defined. On the other hand, it is **associative**. If $f : S \to T$ and $g : T \to U$ and $h : U \to V$ then $(h \circ g) \circ f = h \circ (g \circ f)$.

Each set S (including the empty set) comes equipped with the **identity** function:

$$1_S : S \to S; \text{ defined by } 1_S(s) = s \text{ for all } s \in S$$

Functions that are both injective (one-to-one) and surjective (onto) have inverses. If $f : S \to T$ is such a function (i.e f is a bijection), then the inverse function $f^{-1} : T \to S$ is also a bijection, and:

$$f^{-1} \circ f = 1_S : S \to S \text{ and } f \circ f^{-1} = 1_T : T \to T$$

In this sense, f^{-1} is a **two-sided inverse** of the function f.

We might visualize the category of sets as a *directed graph*. The vertices of the graph correspond to the sets S and arrows of the graph correspond to the functions (it's a very big directed graph!). The identity functions are visualized as arrows that loop from S back to S, and the composition law is an assignment of a new arrow when the tip of one arrow points to the base of another.

The key features of sets stressed above are combined in the following:

Definition 0.1. A **category** $\mathcal{C}$ consists of the following:

• A collection of objects, usually denoted with capital letters, e.g. X, Y, Z.

• A set of morphisms (functions) $\hom(X, Y)$ attached to each pair of objects, whose elements are the morphisms (functions) from X to Y, usually denoted with small letters, e.g. f, g, h.

• A composition operation on morphisms:

$$\circ : \hom(X, Y) \times \hom(Y, Z) \to \hom(X, Z)$$

that is **associative**, and finally:

• An **identity** morphism 1_X in each set $\hom(X, X)$.

Remark. The objects in the categories in this course will all be sets with some additional attributes, and the morphisms we consider will all be functions that "respect" the additional characteristics.

Definition 0.2. (a) An element $f \in \hom(X, Y)$ within a given category $\mathcal{C}$ is an **isomorphism** if f has a two-sided inverse $g \in \hom(Y, X)$. That is:

$$g \circ f = 1_X \text{ and } f \circ g = 1_Y$$

(b) A **symmetry** of X in $\mathcal{C}$ is an isomorphism from X to itself.

Example. If S is a finite set, then the symmetries of S (in the category of sets) are the *permutations* of the elements of S. A set with n elements has $n!$ symmetries.

So how does this related to the colloquial notion of symmetry?

The *plane* $\mathbb{R}^2$ is an example of a set with additional attributes. One of these attributes is the **distance** between two points, given by the Pythagorean Theorem:

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

This makes $\mathbb{R}^2$ into a **metric space**, and the functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ that do not change the distances between points:

$$d(f(x_1, y_1), f(x_2, y_2)) = d((x_1, y_1), (x_2, y_2))$$

are the *Euclidean* symmetries (or congruences) of the plane. In Euclidean geometry, two shapes (e.g. triangles) are said to be congruent if there is a symmetry of the plane that takes one shape to the other. There are analogues for $\mathbb{R}^3, \mathbb{R}^4$, etc.

In the colloquial notion of symmetry, a shape (or pattern) is placed in the plane, and the Euclidean symmetries that preserve the object/pattern are the symmetries of that object/pattern. For example, the 8 symmetries of a square comprise the *dihedral group* of order 8. Each regular polygon has a dihedral group of symmetries attached to it. In dimension three, each of the three pairs of dual Platonic solids has a group of symmetries that we will also study.

Symmetries form a **group**, which can be thought of as a set with the additional attribute of an associative operation (and an identity and inverses). A group is **abelian** if the operation is also commutative. A **vector space** is an abelian group with the additional attribute of *scalar multiplication* by the elements of a specified field of *scalars* (e.g. the real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$). The symmetries of a vector space of dimension n are the *general linear group*, which can be identified with $n \times n$ invertible matrices, and the central problem of representation theory is the following. Given a group G (of symmetries), what are the distinct ways of representing the elements of G as $n \times n$ matrices so that the group operation becomes matrix multiplication?

We will look at differential calculus in one variable and then several variables, and contrast this with calculus in one *complex* variable. The notion of a continuous function is folded into the category of **topological spaces**, and the notion of a differentiable function is folded into the category of (differentiable) **manifolds**. This will lead us to consider groups (of symmetries) that are also manifolds, and to the idea of a **Lie group**, which is vital to understanding particle physics. To find the representations of such groups, we ultimately turn to **algebraic geometry** and homogeneous manifolds of flags which, in a sense, bring us full circle back to sets.

Coming Attractions*

§1 Sets and subsets. Permutations (even and odd).

§2 Abelian groups. Number Theory. Quotients. Classification.

§3 Metric Spaces. The Euclidean Groups and Platonic Solids.

§4 Vector Spaces. Characteristic Polynomials. Jordan normal form.

§5 Inner Product Spaces. Gram-Schmidt. Orthogonal transformations.

§6 Groups. Normal subgroups. Group actions.

§7 Group Representations. Classsification of finite group reps.

§8 Character Tables.

§9 One-Variable Calculus. The real numbers. Differentiable functions.

§10 The Complex Derivative.

§11 Multi-Variable Calculus. Differentiability in more variables.

§12 Topological Spaces. The Fundamental Group.

§13 Manifolds. Sheaves of functions.

§14 Topological Groups and Lie Groups. The Circle and SU(2).

§15 Lie Algebras.

§16 Tensor Products.

§17 Algebraic groups.

§18 Polynomial rings and algebraic varieties.

§19 Line Bundles.

§20 Flags (and sets).

§21 Representations of complex Lie groups.

* Subject to frequent revision.