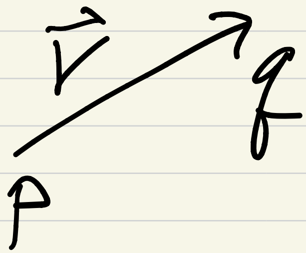
4800-7

Symmetries of $(\mathbb{R}^n, d)$

Key point:



$d(p, q)$

$\vec{v} = q - p$

$$d(p,q)^2 = |\vec{v}|^2 = \vec{v} \cdot \vec{v}$$

dot product.

Symmetries of $\mathbb{R}$:

$$f(x) = \begin{cases} r + x & \text{or} \\ r - x \end{cases}$$

Mimic this for $\mathbb{R}^n$

$$f(\vec{x}) = \begin{cases} \vec{r} + \phi(\vec{x}) \\ \end{cases}$$

where $\phi(\vec{x})$ is an orthogonal (linear) transformation.

An orthogonal transform
is given by a matrix

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$$

columns

s.t. $|\vec{v}_i| = 1$ and $\vec{v}_i \cdot \vec{v}_j = 0$

if $i \neq j$.

$\vec{v}_1, \ldots, \vec{v}_n$ are an orthonormal basis

for $\mathbb{R}^n$

Let:

$$f(x_1,\dots,x_n) = A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$f(x_1,\dots,x_n) = x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n$$

$$\underset{A}{=}$$

Check that f preserves d.

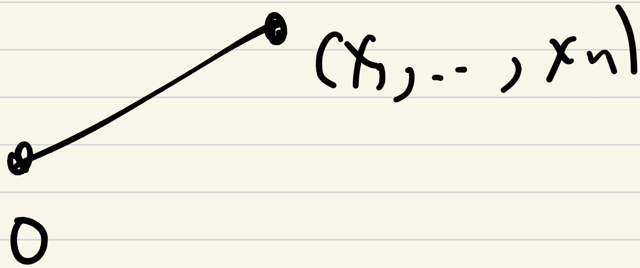
$$|(x_1,\dots,x_n) - (y_1,\dots,y_n)| \overset{?}{=}$$

$$|f(x_1,\dots,x_n) - f(y_1,\dots,y_n)|$$

$$\left|(x_1 - y_1, \dots, x_n - y_n)\right|$$

x's

$$= \left|f(x_1 - y_1, \dots, x_n - y_n)\right|$$



But:

$$f(x_1, \dots, x_n) = x_1\vec{v}_1 + \dots + x_n\vec{v}_n$$

and

$$(x_1\vec{v}_1 + \dots + x_n\vec{v}_n) \cdot (x_1\vec{v}_1 + \dots + x_n\vec{v}_n)$$

$= x_1^2 + \cdots + x_n^2$ ✓

So: $A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is

a symmetry:

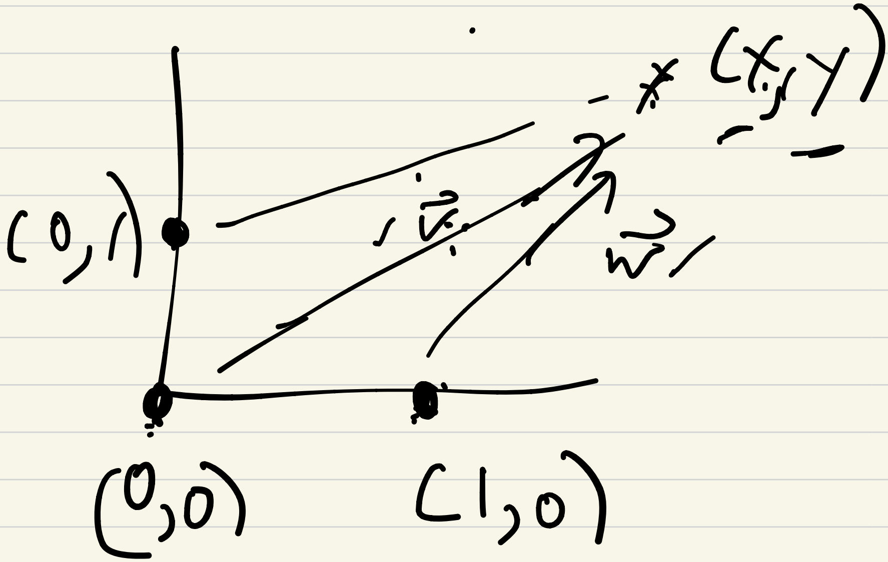
Claim: If

ϕ is a symmentry of $\mathbb{R}^n$

and $\phi(0) = 0$

$\phi(e_1) = e_1, \ldots, \phi(e_n) = e_n$

then ϕ = identity //



suppose $\phi(0)=0$, $\phi(1,0)=(1,0)$.

$|\vec{v}|^2 = x^2+y^2 = d^2$

$|\vec{w}|^2 = (x-1)^2+y^2 = e^2$ //

$-2x+1 = d^2-e^2$ // $x = \frac{e^2-d^2+1}{2}$

Game: Start with

$f: \mathbb{R}^n \to \mathbb{R}^n$ **symmetry**

Let $\underline{f(0) = \vec{r}}$

Then

$$g = \left(T_{-\vec{r}} \circ f\right)(0) = 0$$

Let $g(e_1) = \vec{v}_1, \ldots, g(e_n) = \vec{v}_n$

Then $(A^{-1} \circ g) = id.$

$$A^{-1} \circ \tau_{-\vec{r}} \circ f = id$$

$\boxed{A}$ $\quad \tau_{-\vec{r}} \circ f = A$

$\boxed{\tau_{\vec{r}}}$ $\quad f(x_1, \dots, x_n) = \vec{r} + A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

In $\mathbb{R}^2$: There a lots

of symmetries.

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

θ

O

$$\begin{bmatrix} c & s \\ s & -c \end{bmatrix}$$

$\theta/2$

O

✓

rot

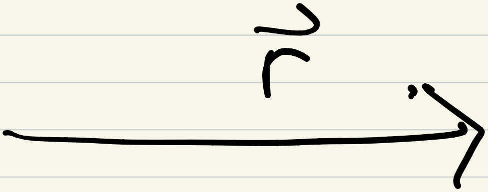
θ

ref

P

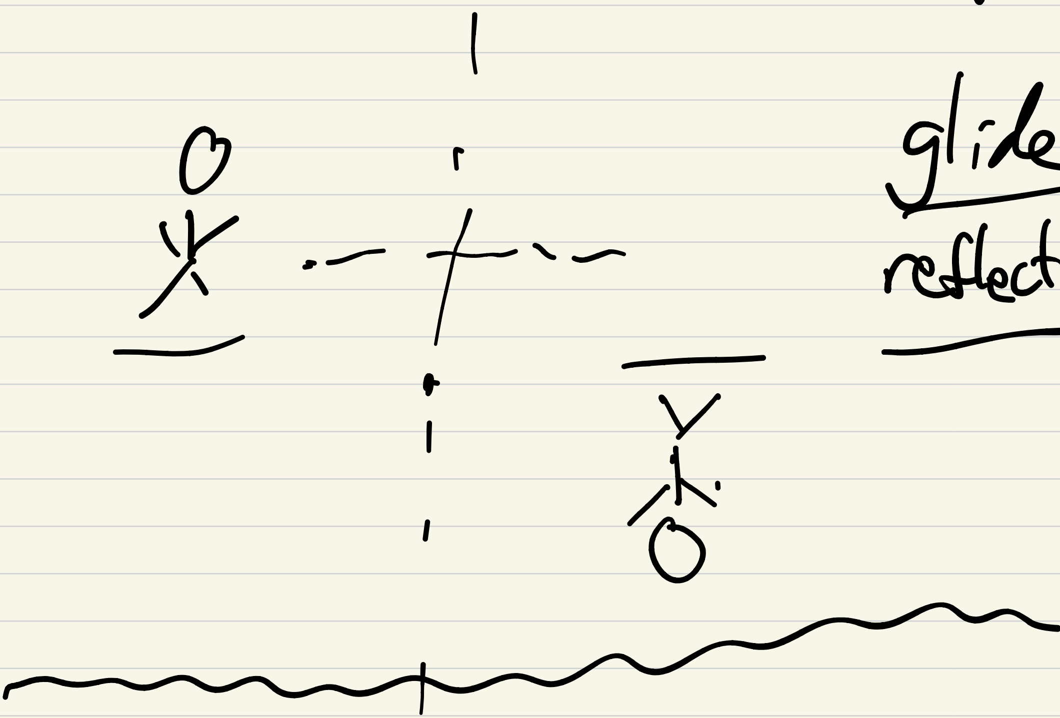
A

O

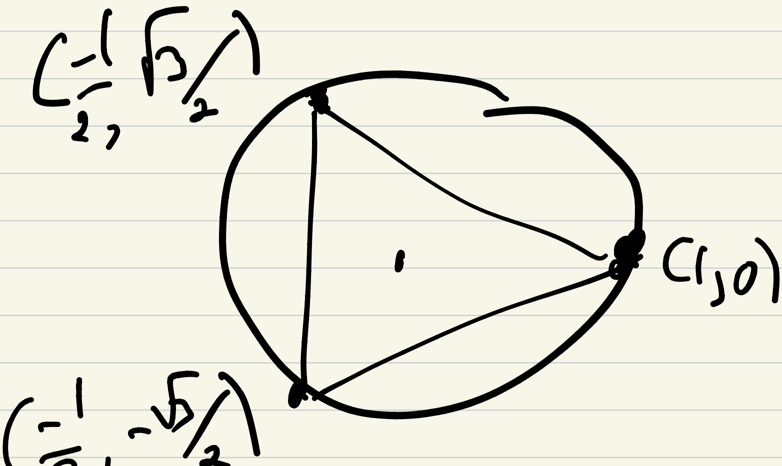


translation

glide reflection



Symmetries of regular n-gons



inscribed in S^1

$e^{2\pi i/n}$

Vertex at: $(1,0)$

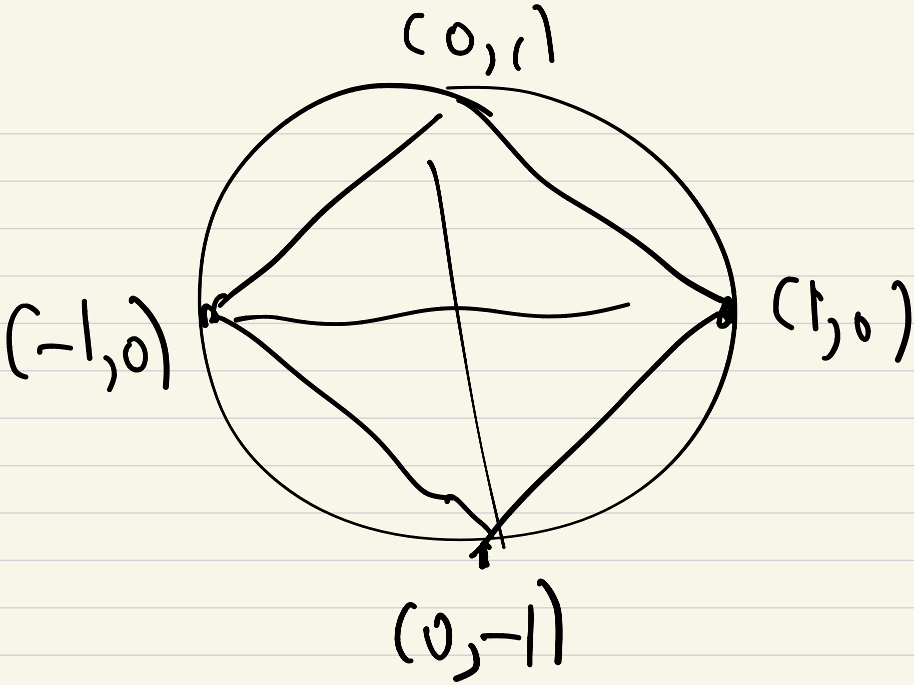
$\parallel$

other vertices: $\left(\cos\left(\frac{2\pi}{n}\right), \sin\left(\frac{2\pi}{n}\right)\right)$

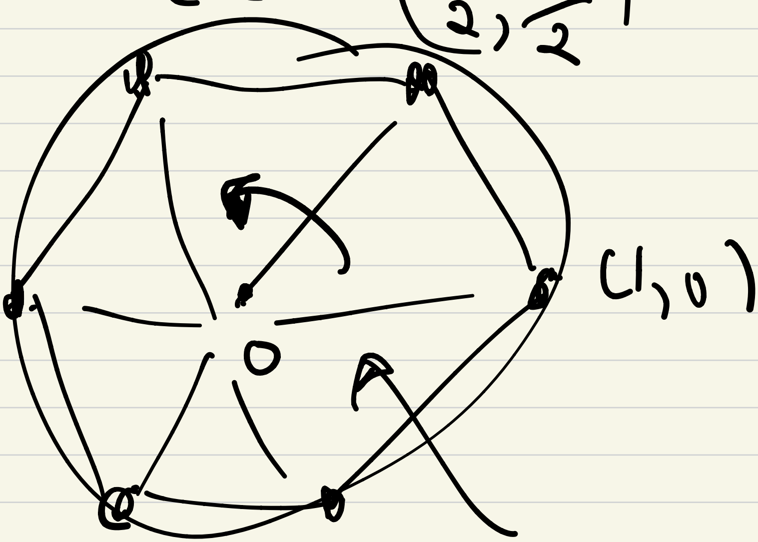
$\vdots$

$\left(\cos\left(\frac{4\pi}{n}\right), \sin\left(\frac{4\pi}{n}\right)\right)$

$\vdots$



etc.

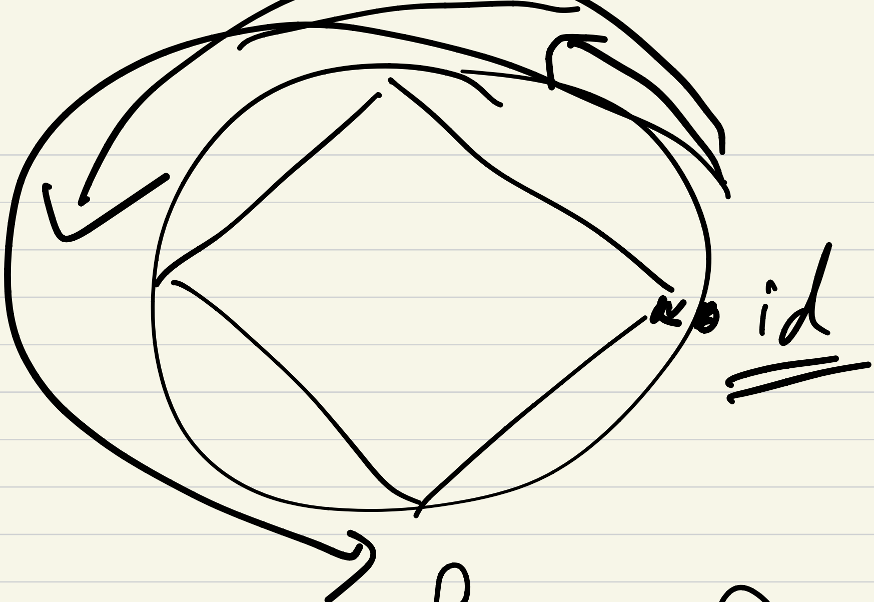


Symmetries of P_n :

Rotations : ϕ_θ

Reflections: ρ_θ

$$\theta = 0, \frac{2\pi}{n}, \ldots, \frac{2\pi(n-1)}{n}$$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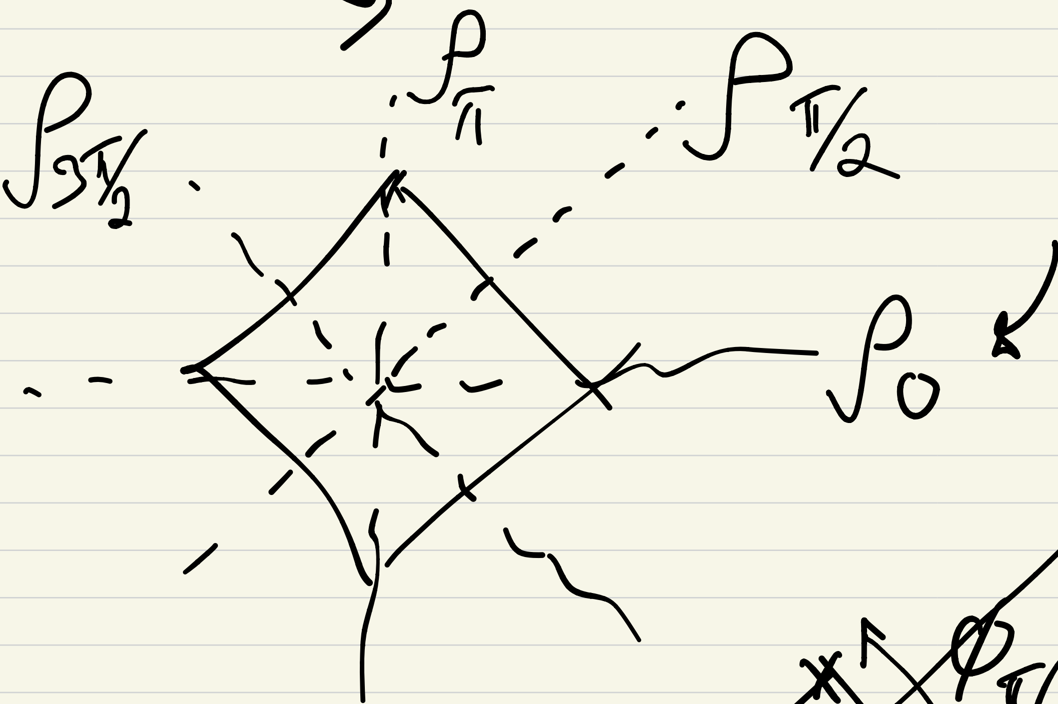
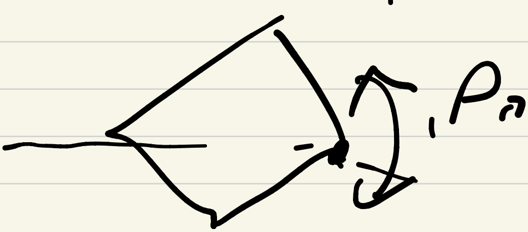
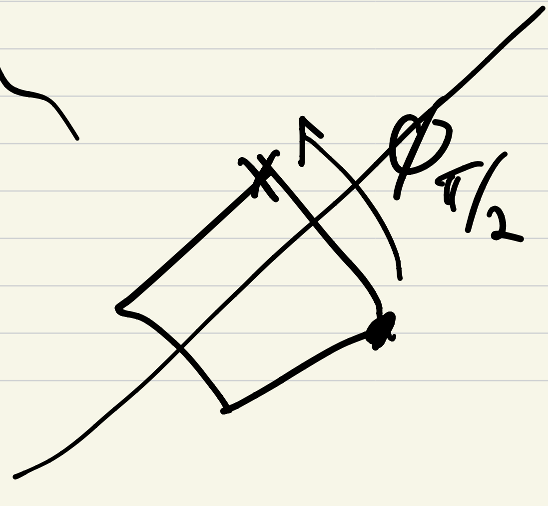

ρ_π
$\rho_{\pi/2}$
$\rho_{3\pi/2}$
ρ_0


ρ_π


$\phi_{\pi/2}$

The rotations of P_n:

are a cyclic gp C_n

$$C_n = \{1, x, x^2, \dots, x^{n-1}\}$$

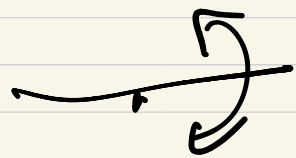
$$x = \phi_{2\pi/n}$$

$$D_{2n} = \left\{ \begin{array}{l} 1, x, x^2, \dots, x^{n-1} \\ y, xy, x^2y, \dots, x^{n-1}y \end{array} \right\}$$

$y = \rho_0$, $xy = \rho_{\pi/2}$ $\qquad y \cdot x$

$$X^k = \begin{bmatrix} c(\theta) & -s(\theta) \\ s(\theta) & c(\theta) \end{bmatrix}$$

$$Y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \rho_0$$



$$X^k Y = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

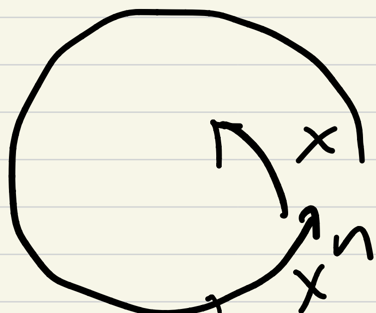
$$= \begin{bmatrix} c & s \\ s & -c \end{bmatrix} = \rho_{\theta/2}$$

To nail down symmetries

- Find generators:

$$x = \phi_{2\pi/n}$$

$$y = \rho_0$$



- Relations:

$$x^n = 1$$

$$y^2 = 1$$

$$yx = x^{n-1}y$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{Y} \cdot \underbrace{\begin{bmatrix} c(\theta) & -s(\theta) \\ s(\theta) & c(\theta) \end{bmatrix}}_{X}$$

$$= \begin{bmatrix} c(\theta) & -s(\theta) \\ -s(\theta) & -c(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} c(-\theta) & s(-\theta) \\ s(-\theta) & -c(-\theta) \end{bmatrix}$$

$$\boxed{\begin{matrix} c(-\theta) = c(\theta) \\ s(-\theta) = -s(\theta) \end{matrix}}$$

$$= X^{n-1} Y$$

reflection across $-\theta/2$

$\left\{ \begin{matrix} 1, x, \ldots, x^{n-1} \\ y, xy, \ldots, x^{n-1}y \end{matrix} \right\}$

D_n = gp. generated by

x, y

with relations

$\boxed{x^n = 1}$, $\boxed{y^2 = 1}$,

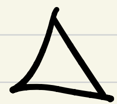
$\boxed{yx = x^{n-1}y} = x^{-1}y^{-1}$

$\boxed{yxyx = 1}$

Game: Find all possible pairs of matrices A, B s.t.

$$\boxed{\vec{A}^n = 1,\ B^2 = 1,\quad BA = A^{-1}B}$$

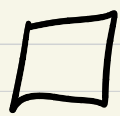
Platonic Solids



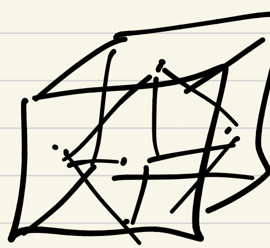
Tetrahedron

(4, 6, 4)

Vert Edges Faces

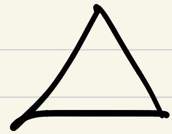


Heptahedron (Cube)



(8, 12, 6)

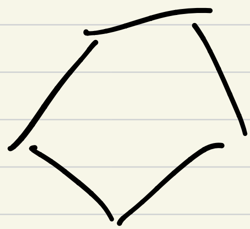
Octahedron:



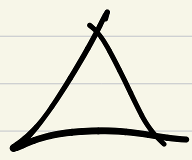
(6, 12, 8)

Dodecahedron:

(20, 30, 12)



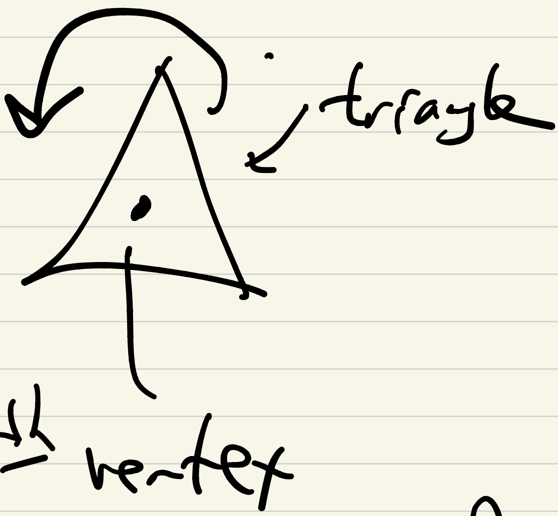
Icosahedron

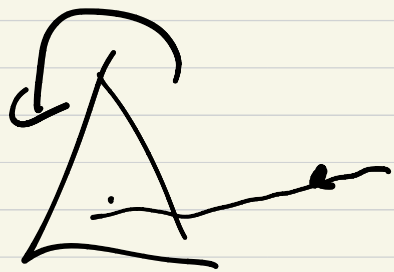


(12, 30, 20)

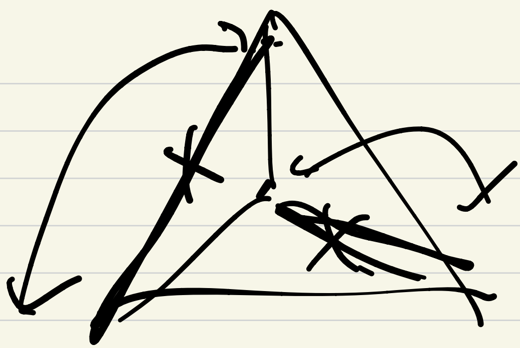
Count orientation-preserving symmetries of the solids:

Thm: Every such symmetry fixes a line through O.

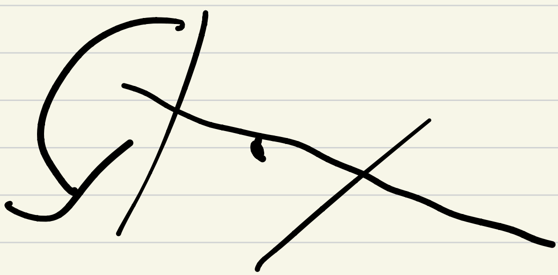




2 rotations × 4 Δ's



Edges in opposite pairs



line of symmetry

3 pairs of opposite edges

$\times$

1 rotation by π

$$1 + 4 \times 2 + 3 \times 1$$

id

rotations around a face (+ opposite vertex)

rotations around opposite edges

$= \underline{\underline{12}}$ = # of even permutations of [4].