Math 4200-001/Complex Analysis/Fall 2017
Second Midterm

1. (20 points) Calculate:

$$\int_\gamma \frac{z}{z^2-1}dz$$

for each of the following paths $\gamma : [0, 2\pi] \to \mathbb{C}$

(a) $\gamma(t) = \frac{1}{2}e^{it}$

(b) $\gamma(t) = \frac{1}{2} + \frac{3}{4}e^{2it}$.

(c) $\gamma(t) = \frac{1}{2} - \frac{3}{4}e^{2it}$.

(d) $\gamma(t) = 2e^{-it}$

2. (10 points) Suppose the polynomial:

$$f(z) = c_d z^d + c_{d-1}z^{d-1} + \cdots + c_0$$

satisfies:

$$|f(z)| \le M \text{ for all points of the circle } |z| = R$$

Find an upper bound for the leading term:

$$|c_d|$$

in terms of M and R.

3. (20 points) In your own words, give the full story why a function with one complex derivative at all points of an open set $U \subset \mathbb{C}$ must in fact have infinitely many complex derivatives at all points of U.

4. (20 points) Suppose $z_0, z_1 \in \mathbb{C}$ are distinct points with

$$|z_1 - z_0| = s < r$$

and that $f(z)$ has an isolated singularity at z_1, but is otherwise analytic on $D_r(z_0)$. Prove that the radius of convergence of the power series:

$$f(z) = \sum_{k=0}^{\infty} c_k(z - z_0)^k$$

is s if and only if z_1 is **not** a removable singularity.

5. (10 points) Show that if $f(z)$ is an entire function and:

$$|f(z)| < M|z|$$

for some $M > 0$, then $f(z) = cz$ for some $c \in \mathbb{C}$.

6. (20 points) Give examples of analytic functions with singularities at the integers of each of the following types.

(a) $f(z)$ has a pole of order one at each $n \in \mathbb{Z}$.

(b) $f(z)$ has a pole at each $n \neq 0$ and a removable singularity at 0.

(c) $f(z)$ has a pole at each square integer n^2 all other integers are removable singularities.

(d) $f(z)$ has an essential singularity at each $n \in \mathbb{Z}$.