Math 4200-001/Foundations of Algebra/Fall 2017
Final Exam (10 Problems). Show your work!

1. (20 points) Expand the function:

$$\frac{1}{(1-z)^3}$$

in a power series around $z = 0$.

2. (20 points) Explain the proof of the fundamental theorem of algebra in your own words.

3. (20 points) Find all the zeroes and singularities of the following functions and classify the singularities.

(a)

$$f(z) = \frac{\cos(z) - 1}{e^z - 1}$$

(b)

$$f(z) = e^{1/(z^2-1)}$$

4. (20 points) Where does the function $z^n + 1$ attain its maximum modulus on the closed unit disk $\overline{D_1(0)}$?

5. (20 points) Let $w \in \mathbb{C}$ be a complex number with $|w| < 1$.

(a) Show that the analytic function:

$$f(z) = \frac{z - w}{1 - \overline{w}z}$$

maps the unit disk $D_1(0)$ to the unit disk.

(b) Calculate $f'(w)$, and check that it is real and positive.

(c) If $g(z)$ is **any** analytic function that maps $D_1(0)$ to $D_1(0)$, show that:

$$|g'(w)| \leq f'(w)$$

(Hint: Use Schwarz's lemma)

6. (20 points) Find a harmonic function on the open set:

$$\mathbb{C} - \{[1, 2]\}$$

that has no harmonic conjugate function.

7. (20 points) Find the residues at 0 of the meromorphic functions:

(a) $f(z) = \dfrac{e^z - 1}{z^5}$

(b) $f(z) = \dfrac{1}{\sin(z) - z}$

8. (20 points) Evaluate the following integral.

$$\int_0^{2\pi} \frac{1}{4 - \sin^2(\theta)} d\theta$$

9. (20 points) Evaluate the following integral.

$$\int_0^{\infty} \frac{x^2}{1 + x^6} dx$$

10. (20 points) Find the Fourier transform of

$$f(x) = \frac{1}{ax^2 + bx + c}$$

assuming a, b, c are real numbers and $b^2 - 4ac < 0$. What happens when $b^2 - 4ac \geq 0$?