Math 4030-001/Foundations of Algebra/Fall 2017
Second Midterm

1. (20 points) Consider the field:

$$\mathbb{Q}[\sqrt{3}] = \{r + s\sqrt{3} \mid r, s \in \mathbb{Q}\}$$

(a) How is multiplication in this field defined?

(b) What is the multiplicative inverse of $r + s\sqrt{3}$?

(c) Notice that the pair of integers $(a, b) = (2, 1)$ satisfies $a^2 - 3b^2 = 1$. Prove that if:

$$(2 + \sqrt{3})^n = a + b\sqrt{3}$$

then the pair (a, b) also solves $a^2 - 3b^2 = 1$. Use this idea to find four more pairs (a, b) of positive integer solutions to $a^2 - 3b^2 = 1$.

(d) Find a quadratic polynomial $p(x)$ with integer coefficients that has $4 - \sqrt{3}$ as a root in $\mathbb{Q}[\sqrt{3}]$.

2. (20 points) Every positive real number has a decimal expansion:

$$\alpha = a.d_1 d_2 ...$$

This is usually uniquely determined by α. However, there are cases in which a single α has two such expansions. Explain how this happens and describe all the real numbers that have two expansions.

3. (20 points) Suppose $p(x)$ is a prime polynomial in $\mathbb{Q}[x]$.

(a) Prove that $p(x - r)$ is also prime for all $r \in \mathbb{Q}$.

(b) Give examples of prime polynomials $p(x)$ and $q(x)$ for which:

$$p(x^2) \text{ is prime and } q(x^2) \text{ is not prime}$$

(c) Can $p(x^2 - 1)$ ever be prime in $\mathbb{Q}[x]$? Explain.

(d) Can $p(x)^2 - 1$ ever be prime in $\mathbb{Q}[x]$? Explain.

4. (20 points) Prove that:

(a) Every non-constant polynomial with complex coefficients factors as a product of linear polynomials with complex coefficients.

(b) Every non-constant polynomial with real coefficients factors as a product of linear and quadratic polynomials with real coefficients.

5. (20 points) Factor the following polynomials completely as a product of prime polynomials in $\mathbb{Q}[x]$:

(a) $x^{10} - 1$

(b) $x^{12} - 1$

(c) $x^{16} - 1$

(d) $x^{18} - 1$.