Math 4030-001/Foundations of Algebra/Fall 2017

Appendix A: Binary Arithmetic

The place value for digits was an extraordinary idea that transformed arithmetic. In our ordinary base ten arithmetic, the values are positive and negative powers of 10 depending on the precise location of the digit (left or right of the decimal point), so that, for example:

$$5294 = 5(10^3) + 2(10^2) + 9(10) + 4(1) \text{ and}$$

$$18.352 = 1(10) + 8(1) + 3\left(\frac{1}{10}\right) + 5\left(\frac{1}{10^2}\right) + 2\left(\frac{1}{10^3}\right)$$

The key features of place values in base ten are:

(i) Each digit is between 0 and 9 and

(ii) Place values are powers of 10.

Place values can be applied to any base $n > 1$, with:

(i) Digits that are symbols for the numbers between 0 and $n - 1$ and

(ii) Place values are powers of n.

Examples. (a) Base 5 has digits:

$$0, 1, 2, 3, 4$$

and place values that are powers of 5, so that, for example:

$$423.01 = 4(5^2) + 2(5) + 3(1) + 0\left(\frac{1}{5}\right) + 1\left(\frac{1}{5^2}\right)$$

(the right side of the equation is written in base ten(!))

(b) Base 16 is usually written with digits:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \text{A,B,C,D,E}$$

and in base 16 we can have numbers like:

$$\text{3B.E} = 3(16) + 11(1) + 15\left(\frac{1}{16}\right)$$

Question. Why is there no base one?

We will focus here on base two, aka **binary** arithmetic, in which:

(i) The digits are 0 or 1 (on or off, for a machine).

(ii) The places are powers of 2.

Thus, counting in binary goes as follows:

$$1, 10, 11, 100, 101, 110, 111, 1000, 1001$$

We add multi-digit numbers in binary as usual by carrying:

$$\begin{array}{r} 101 \\ +\quad 1 \\ \hline 110 \end{array} \qquad \begin{array}{r} 101 \\ +\quad 10 \\ \hline 111 \end{array} \qquad \begin{array}{r} 101 \\ +\quad 11 \\ \hline 1000 \end{array}$$

(and there is a lot of carrying in binary!).

We multiply multi-digit numbers as usual:

$$\begin{array}{r} 101 \\ \times \qquad 110 \\ \hline 000 \\ 1010 \\ 10100 \\ \hline 11110 \end{array} \qquad \begin{array}{r} 111 \\ \times \qquad 110 \\ \hline 000 \\ 1110 \\ 11100 \\ \hline 101010 \end{array}$$

which only involves multiplications by 0 or 1 (and lots of carrying!).

Place values after the decimal point in binary are powers of one-half. Thus:

$$0.111 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

and the repeating decimal:

$$0.\overline{1} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

Rational numbers are repeating decimals in binary. For example,

$$\text{one third} = \frac{1}{11} = 0.\overline{01}$$

as you can see by multiplying:

$$(0.\overline{01})(11) = (0.\overline{01}) + (0.\overline{10}) = 0.\overline{1} = 1$$

You find decimal expansions by long dividing, exactly as in base ten.