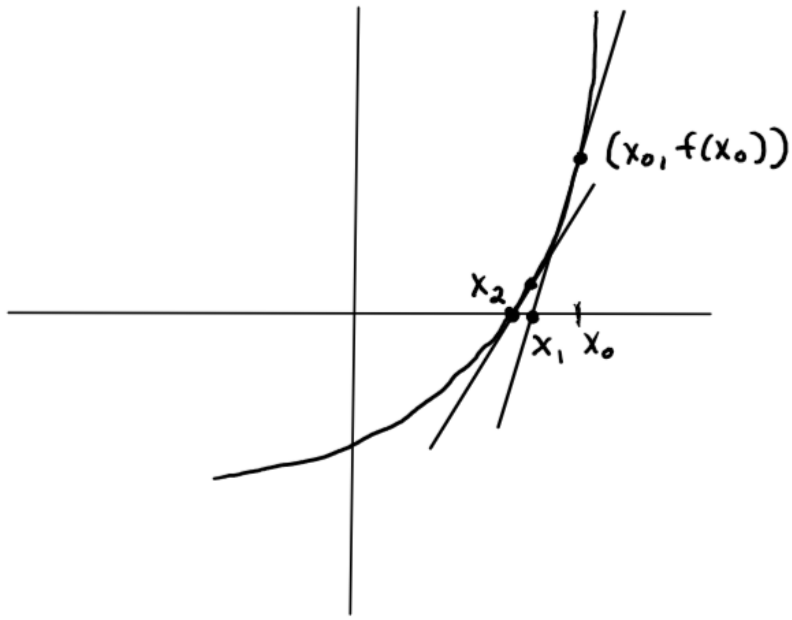
# Newton's Method

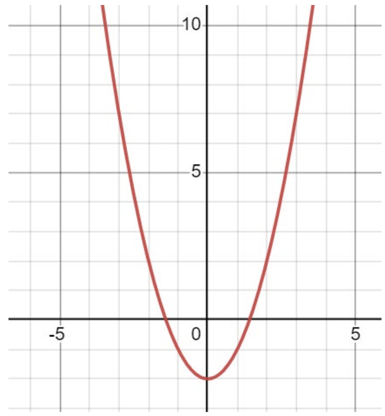
Formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Newton's method is a series of steps to approximate roots of a polynomial.

Start with $x_0$. Plot $(x_0, f(x_o))$ on the graph. Find the tangent line at $x_0$, and draw it on your graph. Let $x_1$ be the intersection of the x-axis and the tangent line. Plot $(x_1, f(x_1))$ and repeat the same steps- giving the result of $x_2$. Keep going until the desired answer is derived. This is typically when the result is stable to the ten-thousandth (4th) decimal place.



Example 1: $f(x) = x^2 - 2$ Actual Root: $x = \sqrt{2} = 1.41421...$




Looking at the graph above, we are going to guess what the positive root of $f(x) = x^2 - 2$ would be → $x_0 = 1$. Then, we will find what f'(x) is → $f'(x) = 2x$.

Guess

$x_0 = 1$

$f(x) = x^2 - 2$

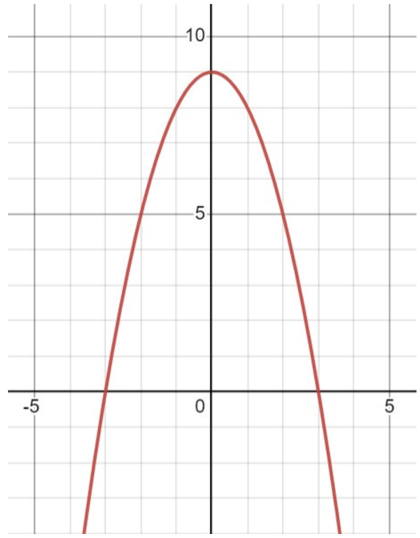
$f'(x) = 2x$

Starting with our guess in the left column of the table, we plug in $x_0$ into $f(x_n)$ in the second column. This gives us $f(x_n)$ = 1. Then, in the third column, we plug $x_0$ into $f'(x_n)$ giving us $f'(x_n)$ = 2. Going to the second row of our first column, we will use our original equation – $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ where n = 0 – to find $x_1$. We repeat the process to find $x_2$, $x_3$, … until we find a number that matches the actual root – $\sqrt{2} = 1.41421...$ up to three decimals.

| $x_n$ | $f(x_n)$ | $f'(x_n)$ |
|---|---|---|
| $x_0 = 1$ | $f(x_0) = (1)^2 - 2 = 1$ | $f'(x_0) = 2(1) = 2$ |
| $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - (-\frac{1}{2}) = \frac{3}{2}$ | $f(x_1) = (\frac{3}{2})^2 - 2 = \frac{1}{4}$ | $f'(x_1) = 2(\frac{3}{2}) = 3$ |
| $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{3}{2} - (\frac{\frac{1}{4}}{3}) = \frac{17}{12}$ | $f(x_2) = (\frac{17}{12})^2 - 2 = \frac{1}{144}$ | $f'(x_2) = 2(\frac{17}{12}) = \frac{34}{12}$ |
| $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{17}{12} - (\frac{\frac{1}{144}}{\frac{34}{12}}) = 1.4142$ | | |

We can see that $x_3$ matches our original root up to 3 decimals.

Example 2: $f(x) = -x^2 + 9$ Actual Root: x = 3




Guess

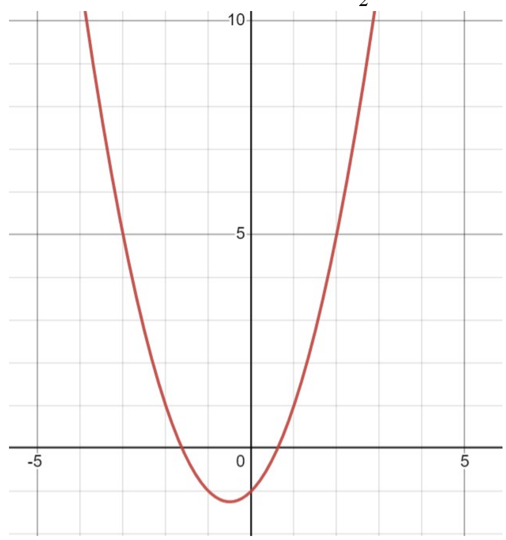
$x_0 = 3.5$

$f(x_n) = -x^2 + 9$

$f'(x_n) = -2x$

| $x_n$ | $f(x_n)$ | $f'(x_n)$ |
|---|---|---|
| $x_0 = \frac{7}{2}$ | $f(x_0) = -(\frac{7}{2})^2 + 9 = -\frac{13}{4}$ | $f'(x_0) = -2(\frac{7}{2}) = -7$ |
| $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{7}{2} - (\frac{-\frac{13}{4}}{-7}) = \frac{85}{28}$ | $f(x_1) = -(\frac{85}{28})^2 + 9 = -0.2155$ | $f'(x_1) = -2(\frac{85}{28}) = -\frac{170}{28}$ |
| $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{85}{28} - (\frac{-0.2155}{-\frac{170}{28}}) = 3.0002$ | | |

Example 3: $f(x) = x^2 + x - 1$ Actual Root: $x = \frac{-1 \pm \sqrt{5}}{2} = 0.618...$ (Golden Mean)




Guess

$x_0 = 0$

$f(x) = x^2 + x - 1$

$f'(x) = 2x + 1$

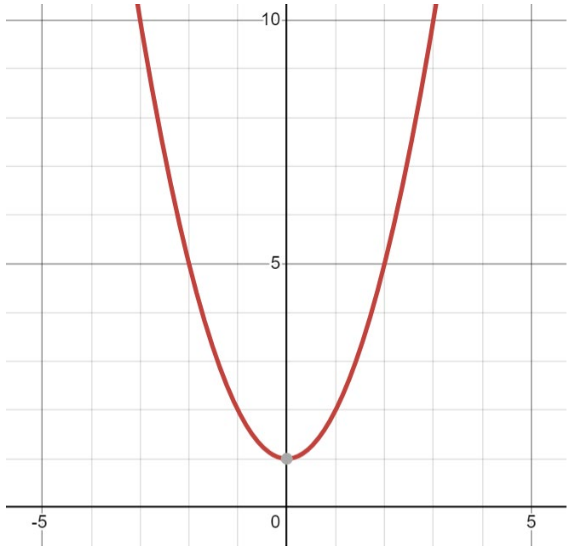
| $x_n$ | $f(x_n)$ | $f'(x_n)$ |
|---|---|---|
| $x_0 = 0$ | $f(x_0) = (0)^2 + 0 - 1 = -1$ | $f'(x_0) = 2(0) + 1 = 1$ |
| $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - (\frac{-1}{1}) = 1$ | $f(x_1) = (1)^2 + 1 - 1 = 1$ | $f'(x_1) = 2(1) + 1 = 3$ |
| $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - (\frac{1}{3}) = \frac{2}{3}$ | $f(x_2) = (\frac{2}{3})^2 + \frac{2}{3} - 1 = \frac{1}{9}$ | $f'(x_2) = 2(\frac{2}{3}) + 1 = \frac{7}{3}$ |
| $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{2}{3} - (\frac{\frac{1}{9}}{\frac{7}{3}}) = \frac{13}{21}$ | $f(x_3) = (\frac{13}{21})^2 + \frac{13}{21} - 1 = \frac{1}{441}$ | $f'(x_3) = 2(\frac{13}{21}) + 1 = \frac{47}{21}$ |
| $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = \frac{13}{21} - (\frac{\frac{1}{441}}{\frac{47}{21}}) = 0.6180$ | | |

In this example, we see a connection between Newton's Method and The Fibonacci Numbers. The value for $x_1 = \frac{1}{1}$ which is the first Fibonacci Number divided by the second, $x_2 = \frac{2}{3}$ is the third Fibonacci Number divided by the fourth, and $x_3 = \frac{13}{21}$ is the seventh number divided by the eighth. So, we see a pattern which helps us predict what $x_n$ will be. Each subsequent $x_n$ is a division of two consecutive Fibonacci Numbers, which numbers increase by powers of 2 as $n$ increases.

$$\frac{1}{1}, \frac{2}{3}, \frac{13}{21}, \frac{610}{987}, \ldots \text{ corresponds to } \frac{1st}{2nd}, \frac{3rd}{4th}, \frac{7th}{8th}, \frac{15th}{16th}, \ldots$$

$$+2^1 \quad +2^2 \quad +2^3 + \ldots.$$

Example 4: $f(x) = x^2 + 1$ Actual Root: none




<u>Guess</u>

$x_0 = 2$

$f(x) = x^2 + 1$

$f'(x) = 2x$

| $x_n$ | $f(x_n)$ | $f'(x_n)$ |
|---|---|---|
| $x_0 = 2$ | $f(x_0) = (2)^2 + 1 = 5$ | $f'(x_0) = 2(2) = 4$ |
| $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - (\frac{5}{4}) = \frac{3}{4}$ | $f(x_1) = (\frac{3}{4})^2 + 1 = \frac{25}{16}$ | $f'(x_1) = 2(\frac{3}{4}) = \frac{3}{2}$ |
| $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{3}{4} - (\frac{\frac{25}{16}}{\frac{3}{2}}) = -\frac{7}{24}$ | $f(x_2) = (-\frac{7}{24})^2 + 1 = \frac{625}{576}$ | $f'(x_2) = 2(-\frac{7}{24}) = -\frac{7}{12}$ |
| $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -\frac{7}{24} - (-\frac{\frac{625}{576}}{\frac{7}{12}}) = -2.1518$ | | |

We see that when we have no real root for a given polynomial, Newton's Method produces values that bounce back and forth in a chaotic manner. For Newton's Method to work as it's meant to, we need the root to exist and for our guess to be close to the actual root.