Infinite decimal expansions of rational numbers

Terminating decimal - if you multiply it by a multiple of ten you get a whole number
Repeating decimal - is a decimal number with a digit, or group of digits, that repeat on and on, without end
Every fraction repeats
Notice that we have repeating decimals made up of the same digits. When you do $\frac{1}{7}$ you’re going through every possible remainder and shifting the decimal result according to the first remainder.

$\frac{1}{11}=.0909$
$\frac{2}{11}=.1818$

There are only two possible remainders when going through long division in either case so the repeating decimal has a repeating block of two digits.

**All rational numbers will repeat!**

Observe the rational number $\frac{5}{13}$. Because it can be written in the form $\frac{p}{q}$, where $q$ does not equal zero, we know this will repeat eventually

$\frac{5}{13}=0.3846153$
$\frac{6}{13}=0.461538$

When we increase the numerator and keep the same denominator, the repeating decimal will shift over beginning with that remainder

\[
\phi(n) = \begin{cases} 1, \phi(2), 3, \phi(5), \phi(7), \phi(13) \\ \phi(2), \phi(7), \phi(13) \end{cases}
\]

\[
\phi(14) = \begin{cases} 1,2,3,4,5,6,7,8,9,10,11,12,13 \end{cases}
\]

\[
\phi(2,7) = \phi(2), \phi(7) = 6
\]

\[
\phi(27) = \phi(3^3) = 3^3 - 3 = 18
\]

**Euler’s theorem to find the length of a repeating block**

If we have a number $n$

$\phi(n)$= number of remainders that are relatively prime to $n$
These will satisfy $\gcd(r,n)=1$
The number of relatively prime numbers will be the repeating block length

$\phi(14) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$, here there are 6 numbers relatively prime to 14 so the repeating block of a fraction with denominator 14 will be 6 digits long

Fermat's little theorem says the length of the repeating block will be $10^{p-1}$ when $p$ is a prime

When $p=7$, repeat after 6 places

$p=11$, repeat after 10 places
10 \div_{13} -3 = 10
10^2 = 9 \equiv (-3)^2
10^3 = 9(-3) = -27 \equiv_{13} -1 \equiv_{13} 12
10^4 = 1 \equiv 9^2 = 3
10^5 = 9 \equiv 3 \equiv -9 \equiv_{13} 4
10^6 = -9 -3 \equiv_{13} 1
10^7 \equiv_{13} 10

100 = 7.13 + 9

1000 = 76.13 + 12

ect.