Math 4030
Linear Equations in More than One Variable
SCRIBE
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For two variables: You will have the equations in the form $(a_1x_1 + a_2x_2)$ & $(b_1x_1 + b_2x_2)$
Where we want to find the values of both x_1 and x_2

There are 3 different types of solutions that you can have when solving a system of two equations with two variables.

- One Solution
- Infinite Solutions
- No solutions (Parallel on a graph)

Before we begin to solve a system with two variables, we can rearrange each equation into slope intercept form, $y = mx + b$. If the two equations have the same slope, we know these lines will be parallel and there will be no solution to the system.

There are also 3 main ways to solve multi variable linear equations:

- Substitution
- Elimination
- Row Reduction in Matrices

Steps to solve Elimination:

- Step 1: Arrange the equations in the standard form: $ax + by + c = 0 \; or \; ax + by = c$
- Step 2: Check if adding or subtracting the equations would result in the cancellation of a variable.
- Step 3: If not, multiply one or both equations by either the coefficient of x or y such that their addition or subtraction would result in the cancellation of any one of the variables.
- Step 4: Solve the resulting single variable equation.
- Step 5: Substitute it in any of the equations to get the value of another variable.

Example (Using Elimination)

$2x + 5y = 20$
$3x + 6y = 12$

Multiply by 3 on the top equation and 2 on the bottom as it will give an LCM for the x coefficient

$(2x + 5y = 20)(3) \rightarrow 6x + 15y = 60$
$(3x + 6y = 12)(2) \rightarrow 6x + 12y = 24$

Subtracting the two equations we get:

$3y = 36$
So $y = 13$

We then plug the y-value back into one of the original equations and we get:
$2x + 5(13) = 20 \rightarrow 2x + 60 = 20 \rightarrow x = -20$

Steps to solve using Substitution:

- Step 1: Solve one of the equations for one variable.
- Step 2: Substitute this in the other equation to get an equation in terms of a single variable.
- Step 3: Solve it for the variable.
- Step 4: Substitute it in any of the equations to get the value of another variable.

Example (Using Substitution)

$x + y = 5 \rightarrow$ Solving for one variable (y) we get $y = 5 - x$
$x - y = 3$

We then plug in y to the second equation: $x - (5 - x) = 3 \rightarrow 2x - 5 = 3 \rightarrow 2x = 8 \rightarrow x = 4$
With $x = 4$ then plugging it into one of the equations (the second one) gives:
$4 - y = 3 \rightarrow y = 1$
So our solution is $(4\,,1)$

Row Operations in Matrices (Using first example):

$\begin{pmatrix} 2 & 5 \\ 3 & 6 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 20 \\ 12 \end{pmatrix}$

Combining into one Matrix we get:

$\begin{pmatrix} 2 & 5 \\ 3 & 6 \end{pmatrix} | \begin{pmatrix} 20 \\ 12 \end{pmatrix}$ Taking $\frac{1}{3}R2$ We get $\begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix} | \begin{pmatrix} 20 \\ 4 \end{pmatrix}$

Then we want to switch R1 and R2 to get $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} | \begin{pmatrix} 4 \\ 20 \end{pmatrix}$ Taking $R2 = R2 - 2(R1)$ we get

$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} | \begin{pmatrix} 4 \\ 12 \end{pmatrix}$ Then by taking $R1 = R1 - 2(R2)$ we get our final matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} | \begin{pmatrix} -20 \\ 12 \end{pmatrix}$

Applications of Linear Equations in More Than One Variable:

- To set up and solve mixture problems in Chemistry

- And to set up and solve uniform motion problems (distance problems)
- Finding optimal rations of dependence for companies

Row Reduction Method for 3 Variables:

Given a system of linear equations;

$x + 2y + 3z = 6$
$2x - 3y + 2z = 14$
$3x + y - z = -2$

Combining into one Matrix we get:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{pmatrix} | \begin{pmatrix} 6 \\ 14 \\ -2 \end{pmatrix} \text{ R2=R2-2(R1) } \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 3 & 1 & -1 \end{pmatrix} | \begin{pmatrix} 6 \\ 2 \\ -2 \end{pmatrix}$$

$$\text{R3=R3-3R1 } \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 0 & -5 & -10 \end{pmatrix} | \begin{pmatrix} 6 \\ 2 \\ -20 \end{pmatrix} \text{ R3=}-\tfrac{1}{5}\text{R3 } \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 0 & 1 & 2 \end{pmatrix} | \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}$$

$$\text{R2}\leftrightarrow\text{R3 } \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -7 & -4 \end{pmatrix} | \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} \text{ R3=R3+7R2 } \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 10 \end{pmatrix} | \begin{pmatrix} 6 \\ 4 \\ 30 \end{pmatrix}$$

$$\text{R3=}\frac{1}{10}\text{R3 } \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} | \begin{pmatrix} 6 \\ 4 \\ 3 \end{pmatrix}$$

So we can restructure the matrix back into a set of equations:

$x + 2y + 3z = 6$
$y + 2z = 4$
$z = 3$

So $z = 3$ and plugging that into the second equation we get $y + 2(3) = 4$ so $y = -2$
And plugging z and y into the first equation we get; $x + 2(-2) + 3(3) = 6$ where $x = 1$
Which gives us our answer of $(1, -2, 3)$

Solving Linear Equations in 3 Variables
$2x - 3y + 4z = -19$
$-5x + 2y + z = -2$
$x + y - 6z = 4$

Steps Using Elimination

1. Add two of the equations to eliminate one variable
2. Add two different equations to eliminate the same variable
3. We're left with two equations with two variables, solve for both variables as we did in a two-variable example
4. Plug in the two variables we have found into any of the original equations and solve for the remaining variable.
5. Then we have a solution of (7, 15, 3)

We can also use substitution and matrices to solve a three-variable system.

Where can we find this information in the Utah Curriculum?

In the curriculum, students do start solving linear equation in one variable in middle school. Starting in 8th grade, students will solve systems of two linear equations in two variables and represent these systems as pairs of lines in the plane. They will find whether these lines intersect, they are parallel, or they are the same line. Students will use linear equations, systems of linear equations, and the slope of a line to analyze situations and solve problems. It is when the students hit high school that they dive further into the method of solving linear equations in multi-variables using substitution and elimination. We see this in standards A.REI.5 & A.REI.6

It further goes into depth in A.REI.10 for understanding that the graph of an equation in two variables is the set of its solutions. While solving systems of linear equations up to three variables using matrix row reduction occurs in standard N.VM.13