Math 4030-001/Foundations of Algebra/Fall 2017

Foundations of the Foundations: Logic

A **mathematical statement** is a sentence with a precise meaning, which is either **true** or **false** (but not both). Examples include:

- $0 = 1$ (False)
- There are finitely many prime numbers. (False)
- The sum of the measures of the interior angles of a triangle in Euclidean space is π. (True)
- The set of rational numbers is countable. (True)
- Every polynomial has a root. (False! Constants have no roots.)

Examples of statements that are not mathematical include:

- Algebra is hard. (No precise meaning.)
- $x = 1$ (x isn't specified)
- This statement is true. (Both true and false!)
- This statement is false. (Neither true nor false!)

Context matters. If, for example, it is clear from the context that we are considering only triangles in Euclidean space, then the statement "The sum of the measures of the interior angles of a triangle is π" is true, but in the context of hyperbolic space the same statement is false. In the context of an equation that determines the value of x, the statement $x = 1$ is mathematical, since it is either true or false. Until the value of x is pinned down, however, the statement isn't mathematical because its truth depends on the unspecified value of x.

We will use variables p and q to stand for mathematical statements.

Not is used to switch the truth value of a mathematical statement. Thus if p is a true statement, then not p is false, and vice versa. This may be written in the following *truth table*:

p	not p
true	false
false	true

Or, and, if...then and **if and only if** are defined by the table:

p	q	p and q	p or q	if p then q	p if and only if q
true	true	true	true	true	true
true	false	false	true	false	false
false	true	false	true	true	false
false	false	false	false	true	true

If..then tends to cause the most trouble, especially when p is false. For example:

$$\text{if } 1 = 0 \text{ then } q$$

is a true statement **regardless** of whether q is true or false.

Notation: The words defined above have corresponding symbols:

$\neg p$	not p
$p \wedge q$	p and q
$p \vee q$	p or q
$p \Rightarrow q$	if p then q
$p \Leftrightarrow q$	p if and only if q

and parentheses may be used in compound statements, as in arithmetic. There is one "order of operation" convention, namely "not" should take priority over other logical expressions. For example:

$$\neg q \Rightarrow \neg p \text{ should be read as } (\neg q) \Rightarrow (\neg p)$$

This is the **contrapositive** of $p \Rightarrow q$, and it is *logically equivalent* to $p \Rightarrow q$, in the sense that their truth tables are the same. It follows that if we want to **prove** the truth of $p \Rightarrow q$, we may decide instead to prove the truth of the contrapositive and achieve the same result.

The statements:

$$q \Rightarrow p \text{ and } \neg p \Rightarrow \neg q$$

are the **converse** and **inverse** to the statement $p \Rightarrow q$, respectively. They are logically equivalent to each other, but not to $p \Rightarrow q$.

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg q \Rightarrow \neg p$	$q \Rightarrow p$	$\neg p \Rightarrow \neg q$
true	true	false	false	true	true	true	true
true	false	false	true	false	false	true	true
false	true	true	false	true	true	false	false
false	false	true	true	true	true	true	true

Other useful logical equivalences include:

$$\neg(p \wedge q) \text{ is logically equivalent to } \neg p \vee \neg q$$

and

$$\neg(p \vee q) \text{ is logically equivalent to } \neg p \wedge \neg q$$

and

$$\neg(p \Rightarrow q) \text{ is logically equivalent to } p \wedge \neg q$$

and

$$(p \Rightarrow q) \wedge (q \Rightarrow p) \text{ is logically equivalent to } p \Leftrightarrow q$$

all of which you can check with truth tables.

Next, we tackle **Quantifiers.** These occur in the following setting:

• S is a sentence that makes use of a variable x.

• A is a set of values for x, each of which makes S either true or false (in other words, making S into a mathematical statement). Then:

(a) "$(\forall x \in A)\ S$" is the mathematical statement "for all x in A, S" i.e. every value of x taken from the set A makes S true and

(b) "$(\exists x \in A)\ S$" is the statement "there exists $x \in A$ such that S" i.e. some value of x taken from the set A makes S true.

Examples: (i) $(\forall x \in \mathbb{Z})\ -(-x) = x$ states that the additive inverse of the additive inverse of every integer returns to the original integer. This is true.

(ii) $(\exists x \in \mathbb{Q})\ x^2 = 2$ states that there is a rational square root of 2. This is false. If A were the set $\mathbb{R}$ of real numbers, it would be true.

(iii) $(\forall x \in \mathbb{R})\ 1/(1/x) = x$ states that the reciprocal of the reciprocal of every real number is the real number. This is not a mathematical statement because the reciprocal of 0 is not defined. If A were, instead, the set $\mathbb{R}^*$ of real numbers **other** than 0, the statement would be true.

Negating Quantifiers:

"not $(\forall x \in A)S$" is the same as "$(\exists x \in A)$ not S" and

"not $(\exists x \in A)S$" is the same as "$(\forall x \in A)$ not S"

The statement "it is not true that every element of A makes S true" is the same as "it **is** true that **some** element of A makes S false"! Similarly, "it is not true that some element of A makes S true" is the same as "it **is** true that **every** element of A makes S false."

Double Quantifiers: Most interesting mathematical statements use multiple quantifiers. If S is a sentence involving variables x and y, and if A and B are sets such that S becomes a statement whenever x is takes a value in A and y takes a value in B, then two quantifiers may be put together to construct one of the mathematical statements below:

$$(\forall x \in A)(\forall y \in B)\ S, \quad (\forall x \in A)(\exists y \in B)\ S, \quad (\forall y \in B)(\exists x \in A)\ S$$

$$(\exists x \in A)(\exists y \in B)\ S, \quad (\exists y \in B)(\forall x \in A)\ S, \quad (\exists x \in A)(\forall y \in B)\ S$$

The meaning of the the first column of unmixed quantifiers is clear (and is the same if the order of the quantifiers is reversed!). The mixed quantifiers, however, are more tricky to interpret. It is useful to think of them in terms of a game with two players:

"$(\exists x \in A)(\forall y \in B)\ S$" means that there is a value player one can assign to x that forces S to be true no matter what player two does.

"$(\forall x \in A)(\exists y \in B)\ S$" means that for each value player one gives to x, player two can find a value to give to y so that S is true.

The order of quantifiers **does** matter when they are mixed.

Important Example. (i) $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})\ x < y$ is a true statement because for each value for x that player one chooses, the second player can find a value of y so that x is smaller than y.

(ii) $(\exists y \in \mathbb{Z})(\forall x \in \mathbb{Z})\ x < y$ is false. Once player one chooses y, then there are always values of x that are not smaller than y.

Make sure that you understand this example!

Next, we run through some more examples from arithmetic:

- **Commutative Law of Addition of Integers**

$$(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})\ x + y = y + x$$

- **Existence of an Additive Identity Integer**

$$(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z})\ x + y = y$$

(there is exactly one such x, namely $x = 0$).

- **Existence of an Additive Inverse Integers**

$$(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})\ x + y = 0$$

(once x is chosen, there is one such y, namely $y = -x$)

Negating Double Quantifiers. The rule for negating single quantifiers extends straighforwardly to double (and multiple) quantifiers. As the "not" passes across each quantifier, the quantifier "flips". Thus:

$$\neg\ (\forall x \in A)(\forall y \in B)\ S \text{ is equivalent to } (\exists x \in A)\,(\neg(\forall y \in B)\ S)$$

$$\text{which is equivalent to } (\exists x \in A)(\exists y \in B)\ \neg S \text{ and}$$

$$\neg\ (\forall x \in A)(\exists y \in B)\ S \text{ is equivalent to } (\exists x \in A)(\forall y \in B)\ \neg\ S$$

Example. The definition of $\lim_{x \to x_0} f(x) = L$ is:

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})\ |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$$

To show that the limit is **not** L (or doesn't exist), we need to show:

$$(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})\ |x - x_0| < \delta \text{ and } |f(x) - L| \geq \epsilon$$

i.e. there is a fixed ϵ and values of x arbitrarily close to x_0 so that $f(x)$ stays at least ϵ away from the value L. (Don't worry....this is analysis!)

Exercises. 2.1. Verify each of the following with a truth table:

$$\neg(p \wedge q) \text{ is logically equivalent to } \neg p \vee \neg q$$
$$\neg(p \vee q) \text{ is logically equivalent to } \neg p \wedge \neg q$$
$$\neg(p \Rightarrow q) \text{ is logically equivalent to } p \wedge \neg q$$
$$(p \Rightarrow q) \wedge (q \Rightarrow p) \text{ is logically equivalent to } p \Leftrightarrow q$$

2.2. A compound statement is a **tautology** if it is true no matter whether the individual statements are true or false. For example:

$$p \vee \neg p$$

is a tautology because both true and false values for p make it true. Which of the following are tautologies?

(a) $(p \Rightarrow q) \vee (q \Rightarrow p)$

(b) $(p \Rightarrow q) \vee \neg q$

(c) $(p \wedge q) \vee \neg q$

(d) $(p \vee q) \vee \neg q$

(e) $(p \vee q) \vee (\neg p \wedge \neg q)$

2.3. Show, with truth tables, the distributive laws for **and** and **or**:

(a) $p \vee (q \wedge r)$ is logically equivalent to $(p \vee q) \wedge (p \vee r)$

(b) $p \wedge (q \vee r)$ is logically equivalent to $(p \wedge q) \vee (p \wedge r)$.

2.4. Convert each English sentence to a quantified math statement.

(a) The multiplication of integers is commutative.

(b) The multiplication of rational numbers is associative.

(c) There is no real square root of -1.

(d) Every real number has a real cube root.

(e) The real cube root of a real number is unique.

(f) Every real number except zero has a multiplicative inverse.

(g) There is no largest integer.

(h) There is a rational number between any two distinct reals.