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Scribe Notes (10/3): Functions and their Power Series Expansions

A power series is a polynomial with an infinite number of terms. A Taylor series is the value of a function at some point $f(a)$, that we can write as an infinite series. Each term in a Taylor series will be related to the function's derivatives, $f^{(n)}(x)$. A Maclaurin series is a Taylor series that is centered at 0. An example of a Maclaurin series is $e^x = \Sigma \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ...$

<u>Maclaurin Series General Form</u>: $f(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} \cdot x^2 + \frac{f'''(0)}{3!} \cdot x^3 + ...$

Examples of Common Maclaurin Series

Functions	Maclaurin Series	Expanded Form	Notes
Geometric Series: $\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + ...$	This series converges when x is small.
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$	
$ln(1 + x)$	$\sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n}$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$	We can find this from the Geometric Series.

How do we know that $ln(1 + x) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n}$? We know that the derivative of $ln(1 + x)$ is equal to $\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - ...$. So, to find $ln(1 + x)$, we need to integrate $\frac{1}{1+x}$, which gives us $ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$. Now that we know $ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...$, we can see that $ln(1 + 1) = 1 - \frac{1^2}{2} + \frac{1^3}{3} - ...$ converges. In fact, this series converges on the interval (-1,1].

We also know that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. So, if we look at $f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$, we see that $\sum_{n=0}^{\infty} ((-x)^2)^n = 1 - x^2 + x^4 - x^6 + ... = \frac{1}{1+x^2}$, which is the derivative of $arctan(x)$. If we integrate this series, we find $arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$. When x=1, $arctan(1) = 1 - \frac{1^3}{3} + \frac{1^5}{5} - \frac{1^7}{7} + ... = \frac{\pi}{4}$.

Finding a Radius of Convergence

Ratio Test Theorem: Let $P(x) = \sum_{n=0}^{\infty} c_n x^n$ be a power series. Then, $P(x)$ converges at x if $lim\, n \to \infty \left| x\left(\frac{c_{n+1}}{c_n}\right) \right| < 1$.

Example: Does e^x converge?

We know that the Maclaurin series expansion for e^x is $\frac{x^n}{n!}$. Using the Ratio test:

$$lim\, n \to \infty \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$
$$= lim\, n \to \infty \left| \frac{x^{n+1}}{(n+1)n!} \cdot \frac{n!}{x^n} \right|$$
$$= lim\, n \to \infty \left| \frac{(x^n)(x)}{n+1} \cdot \frac{1}{x^n} \right|$$
$$= lim\, n \to \infty \left| \frac{x}{n+1} \right|$$
$$= x \cdot lim\, n \to \infty \left| \frac{1}{n+1} \right|$$
$$= x \cdot 0$$
$$= 0$$

Since $0 < 1$, e^x converges for any x.

Maclaurin Series for $sin(x)$

In order to look at the Maclaurin series for sinx, we need to take derivatives. We know that for e^{cx}, $(e^{cx})' = c \cdot e^{cx}$ and $(e^{cx})'' = c^2 e^{cx}$. When we look at the derivatives of sin(x) when x=0, we see a pattern:

$$sin(0) = sin(0) = 0$$
$$(d/dt)sin(0) = cos(0) = 1$$

$(d/dt)^2 sin(0) =- sin(0) = 0$

$(d/dt)^3 sin(0) =- cos(0) =- 1$

=0→→ continues on 0,1,0,-1,0…

So $sin(x) = x - (x^3)/(3!) + (x^5)/5! -$.... and $cos(x) = 1 - (x^2)/(2!) + (x^4)/(4!) -$.... Adding cos(x) and isin(x) together will give us $cos(x) + isin(x) = 1 + ix + (ix)^2/(2!) + (ix)^3/(3!) +... = e^{ix}$. So we get the following important equation:

★ $e^{it} = cos(t) + sin(t))$

Interesting Equations

In the 1600s, someone noticed that $(\pi/4) = arctan(1) = 4arctan(1/5) - arctan(1/239)$. Before that, Archimedes would use polygons to get the approximate solution of π. As with Taylor polynomials, the more polygons Archimedes used to approximate π, the closer he got to its exact value.

Another equation that a mathematician claimed came to him in a dream is the following:

$$(1/\pi) = (2\sqrt{2})/9801 \cdot \sum_{n=0}^{\infty} [(4n!)(1103 + 26390n)]/[(n!)^4 \cdot (396)^{4n}].$$