Math 4030-001/Foundations of Algebra/Fall 2017 Introduction

This is a course on the Foundations of Algebra, so the first question is: What is Algebra?

There are several answers to this.

Algebra is arithmetic with symbols. This is useful for stating general rules like the *Rules of Exponentiation*:

$$x^m \cdot x^n = x^{m+n}, \ x^n y^n = (xy)^n \text{ and } (x^m)^n = x^{mn},$$

The Pythagorean Theorem: $a^2 + b^2 = c^2$ when a and b are the side lengths of a right triangle and c is the length of the hypotenuse,

The Quadratic Formula: The solutions to $ax^2 + bx + c = 0$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Algebra is manipulating equations to solve for an unknown. E.g.

$$5x + 3 = 7$$

$$5x = 4$$
 (subtracting 3)

$$x = 4/5$$
 (dividing by 5)

or the derivation of the quadratic formula. In **linear** algebra, systems of linear equations are manipulated to solve for several unknowns.

Algebra is the study of properties shared (or not) in different contexts. For example, the composition of functions is an associative operation:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

as are the operations of addition and multiplication of numbers:

$$(x + y) + z = x + (y + z)$$
 and $(xy)z = x(yz)$

and the intersection and union of sets:

$$(A \cap B) \cap C = A \cap (B \cap C)$$
 and $(A \cup B) \cup C = A \cup (B \cup C)$

As another example, the two functions:

 $\log(x)$ and $\deg(f(x))$

the logarithm of a (positive) number and the degree of a (nonzero) polynomial both convert multiplication to addition:

 $\log(xy) = \log(x) + \log(y) \text{ and } \deg(f(x) \cdot g(x)) = \deg(f(x)) + \deg(g(x))$

This shared property enables *Euclid's Algorithm* to find the greatest common divisors of pairs of natural numbers and pairs of polynomials.

Algebra encompasses Number Theory which is the detailed study of the natural numbers. The *Fundamental Theorem of Arithmetic* says every natural number factors as a unique product of *prime* numbers, but it doesn't tell us what the prime numbers are! The distribution of prime numbers is one of the most fascinating problems in mathematics. Truncated arithmetic (mod p) of natural numbers is an important tool in number theory, which can also be applied to polynomials, leading to number *fields*, which are literally *algebras* over the rational numbers and may be studied with the tools of linear algebra.

Algebra also refers to the investigation of the nature of symmetry, as manifested in the study of the *groups of symmetries* that can arise as symmetries of space (Representation Theory), of roots of polynomials (Galois Theory), or of groups themselves (Geometric Group Theory).

We will focus in this class on algebra in the context of:

- Sets, Equivalence Classes and Symmetry in its most basic form
- Logic, Quantifiers and Proofs
- Natural Numbers and Integers
- The Fundamental Theorem of Arithmetic
- Rational Numbers
- Real and Complex Numbers
- Polynomials and Rational Functions
- The Fundamental Theorem of Algebra.
- Truncated Natural Numbers and Truncated Polynomials
- Algebraic and Constructible Numbers
- Symmetries of Roots (Galois Theory)

My goal is to make this as hands-on as possible, and never to stray too far from "Algebra" in the secondary school curriculum. Students are invited to rein me in if I do stray out of an excess of enthusiasm....