Math 3210-4/ Honors Foundations of Analysis/Fall 2016
First Midterm (Due Wed September 28)

1. Prove that the set of rational numbers p satisfying $p^3 < 2$ has a real least upper bound, but that the least upper bound is not rational.

2. Suppose that the ordering $<$ of an ordered set S is replaced with $>$. Is S with this "opposite" ordering still an ordered set? (Why/not?). Now suppose we do the same to an ordered **field** F. Is the field with the "opposite" ordering still an ordered field? Why or why not?

3. A *quaternion* is an ordered triple (a, b, c, d) of real numbers, with quaternionic addition defined via:

$$(a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$$

and quaternionic multiplication defined via the distributive law and:

$$\begin{aligned}(1, 0, 0, 0) \cdot (a, b, c, d) &= (a, b, c, d)\\ (0, 1, 0, 0) \cdot (a, b, c, d) &= (-b, a, -d, c)\\ (0, 0, 1, 0) \cdot (a, b, c, d) &= (-c, d, a, -b)\\ (0, 0, 0, 1) \cdot (a, b, c, d) &= (-d, -c, b, a)\end{aligned}$$

Prove (as was done in the book in Theorem 1.25) that all of the field axioms as listed in Definition 1.12 are satisfied **except** for (M2).

4. Prove Theorem 1.31 of the book (even though it is "quite trivial.")

5. Prove that the Schwarz inequality in Theorem 1.35 is an equality if and only if there is a complex number z such that:

$$(b_1, ..., b_n) = (za_1, ..., za_n)$$

i.e. if and only if the two vectors are "complex colinear."

6. Find a one-to-one and onto map from the open interval $(0, 1)$ to $\mathbb{R}$.

7. Find a subset $E \subset \mathbb{R}$ that has every integer as a limit point and has no other limit points. Is it possible to find a subset $E \subset \mathbb{R}$ that has every rational number as a limit point but has no other limit points?

8. Prove or disprove the following. If $Y \subset X$ is open, then a subset $E \subset Y$ is open relative to Y if and only if E is open (as a subset of X).

9. Let $E \subset X$ be a compact subset of a metric space. Show that E is closed and bounded. Note: You cannot apply Theorem 2.41 since we are not assuming that $X = \mathbb{R}^k$.