Math 2200-002/Discrete Mathematics

Combinatorics

Permutations and Combinations are at the heart of combinatorics. Let $r \leq n$ be natural numbers.

Definition. The quantity $P(n, r)$ is the number of ways of selecting r elements (in order) from a set with n elements.

Examples. (a) The number of ways of assigning gold, silver and bronze medals to a group of 10 runners is $P(10, 3)$.

(b) The number of injective functions f from a set A with r elements to a set B with n elements is $P(n, r)$, since we can think of A as the set $\{1, ..., .r\}$, and then an injective function can be thought of a selection of r elements of B via: $f(1), f(2), ..., f(r)$.

Properties of $P(n, r)$.

(i) $P(n, 1) = n$.

(ii) $P(n, r + 1) = P(n, r) \cdot (n - r)$ whenever $r < n$.

Property (i) is clear. To see property (ii), notice that in order to select $r + 1$ elements, we may first select r elements, and then after we have done so, there are $r - n$ ways in which to select the $r + 1$st element (because there are only $n - r$ elements left to select from). But

$$Q(n, r) = \frac{n!}{(n - r)!}$$

also satisfies properties (i) and (ii), since:

$$\frac{n!}{(n - 1)!} = n \quad \text{and} \quad \frac{n!}{(n - r)!} \cdot (n - r) = \frac{n!}{(n - r - 1)!}$$

and so, by induction (on r for each fixed n), we have:

$$P(n, r) = \frac{n!}{(n - r)!}$$

Notice, however, that we do need to make the convention that $0! = 1$ in order to write $P(n, n)$ in this form.

Example. $P(n, n) = n!$ is the number of ways of ordering the elements of a set with n elements. An ordering can be thought of as the same thing as selecting all the members of the set, one by one.

Definition. The quantity $C(n, r)$ is the number of ways of selecting a subset of r elements from a set with n elements.

Example. The number of ways of giving prizes without specifying first, second and third place to three runners out of 10 is $C(10, 3)$.

Properties of $C(n, r)$.

(i) $C(n, 0) = 1$ and $C(n, n) = 1$.

(ii) $C(n, r) = C(n, n - r)$.

(ii) If $r > 0$, then $C(n, r) = C(n - 1, r - 1) + C(n - 1, r)$.

(iv) $C(n, 0) + C(n, 1) + \cdots + C(n, n) = 2^n$.

Proof. For (i), the only subset of S with no elements is $\emptyset$, and the only subset with the same number of elements as S is S itself.

There is a bijection between (the set of) subsets of S with r elements and (the set of) subsets of S with $n - r$ elements given by:

$$f(T) = S - T$$

This bijection demonstrates the equality (ii).

For (iii), choose $s \in S$, and divvy up the subsets of S with r elements:

- The subsets that do not contain s. There are $C(n - 1, r)$ of these.
- The subsets that do contain s. There are $C(n - 1, r - 1)$ of these.

Thus by adding these numbers, we get (iii).

As for (iv), we have seen that the **total** number of subsets of S is the size of the power set $\mathcal{P}(S)$, which is 2^n. Thus by adding up the numbers of subsets with each number r of elements, we get (iv).

Proposition.

$$C(n, r) = \frac{n!}{r!(n - r)!}$$

Proof 1. (Pure thought) The number $P(n, r)$ counts the ways to **select** r elements out of n, but this produces all subsets, equipped with orderings (remembering the order in which the elements of the subset were chosen). Thus to get the number of subsets, we have to **divide** by all the ways of ordering each subset, which is $P(r, r)$. Thus:

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!}{r!(n - r)!}$$

Proof 2. (Induction) This formula is clearly true for $n = 1$ and $r = 1$. Now suppose it is true for $n-1$ and all values of $r \leq n-1$. Then:

$$C(n,r) = C(n-1,r-1) + C(n-1,r) =$$
$$\frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-1-r)!} =$$
$$\frac{r(n-1)!}{r(r-1)!(n-r)!} + \frac{(n-r)(n-1)!}{r!(n-r)(n-1-r)!} =$$
$$r\left(\frac{(n-1)!}{r!(n-r)!}\right) + (n-r)\left(\frac{(n-1)!}{r!(n-r)!}\right) =$$
$$\frac{n(n-1)!}{r!(n-r)!} = \frac{n!}{r!(n-r)!}$$

The Binomial Theorem. For all natural numbers n,

$$(x+y)^n = C(n,0)x^n + C(n,1)x^{n-1}y + \cdots + C(n,n-1)xy^{n-1} + C(n,n)y^n$$

Examples.

- $(x+y)^1 = x + y$ (since $C(1,0) = C(1,1) = 1$).
- $(x+y)^2 = x^2 + 2xy + y^2$ ($C(2,1) = 2$ and $C(2,0) = C(2,2) = 1$).
- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.
- $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

The coefficients $C(n,r)$ are the nth row of **Pascal's Triangle.**

$$\begin{array}{ccccccccccc} & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

in which each entry is the sum of the two entries above it, which is:

$$C(n,r) = C(n-1,r-1) + C(n-1,r)$$

in a visual format.

Proof 1. (Intuitive) Performing the multiplication $(x+y)(x+y)\cdots(x+y)$, using the distributive law but **not** using the commutative law gives every possible "word" consisting of x's and y's exactly once. Thus, for example:

$$(x+y)(x+y) = xx + xy + yx + yy$$

and

$$(x+y)(x+y)(x+y) = xxx + xxy + xyx + yxx + xyy + xyx + yyx + yyy$$

The set of locations of y variables in one of these words is a subset of $\{1, ..., r\}$. There are $C(n, r)$ subsets with r elements (occupied by y variables), which means that after simplifying with the commutative law, the term $x^{n-r}y^r$ occurs exactly $C(n, r)$ times in the product.

Proof 2. (Induction). We've seen in the Example above that the Theorem is true for $n = 1$. If it is true for n, then multiplying by $(x + y)$ and using the identity:

$$C(n, r) + C(n, r - 1) = C(n + 1, r)$$

gives the Theorem for $n + 1$ since $x^{n+1-r}y^r$ appears with coefficient $C(n, r)$ in $x(x + y)^n$ and with coefficient $C(n, r - 1)$ in $y(x + y)^n$.

More Examples. We can recover one of our properties by:

$$2^n = (1 + 1)^n = C(n, 0) + C(n, 1) + \cdots + C(n, n)$$

and get a new property from:

$$0 = (1 - 1)^n = C(n, 0) - C(n, 1) + C(n, 2) - \cdots$$

(we'll use this in inclusion/exclusion).

Homework 10.

§6.3 (2 points each) # 12, 16,18, §6.4 (1 point each) # 4, 12, 24, 28.