# Lecture Notes in Elementary Fluid Dynamics

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# 1 Forces

Forces make fluids move.

# 1.1 Introduction

There are a few instances (of the order of ten) when Science had the greatest **impact** on people's life. Fluid Dynamics provided at least one of them. The list of greatest impacts could include, for example,

- Telecommunications (Electromagnetic theory)
- Nuclear energy (Quantum mechanics)
- Airplanes (Fluid mechanics)

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What is a fluid? Fluids (unlike solids) can be easily deformed: Small forces (no matter how small) can produce large deformations. A sphere-like blob of fluid can easily become a spagetti-like blob



# Demonstration: Overhead. Transparent glass plate with water. Food-coloring blobs.

Fluids can flow.

Fluids include both liquids and gases.

The large deformations under small forces give rise to essentially nonlinear equations. Nonlinear science mainly grew from Fluid dynamics; in particular, let us mention two intensive developments

- Solitons and Integrability.
- Turbulence.

Richard Feynman: The problem of turbulence is "common for many sciences"; it is "the central problem that people need to solve". He gives two examples of turbulence: (1) water flow in a pipe, (2) the motion of galaxies (with stars instead of molecules).

#### How to describe fluid motion?

$$x(a, b, c, t), \quad y(a, b, c, t), \quad z(a, b, c, t), \quad \text{or} \quad \mathbf{x}(\mathbf{a}, t).$$

This is called the Lagrangian description. Good for solids. Not very practical for fluids (see the last demonstration). More practical

 $u(x, y, z, t), \quad v(x, y, z, t), \quad w(x, y, z, t), \quad \text{or} \quad \mathbf{u}(\mathbf{x}, t).$ 

This defines the velocity field:



This is called the Eulerian description.

#### What makes fluids move? Forces!

Two Types of Forces in Fluids:

1. Long-range forces (decrease slowly with distance).

Examples:

- Gravity
- Electromagnetic forces (when the fluid contains charged particles)
- Inertial forces (when the fluid is considered from a non-inertial frame).

These forces can penetrate into the interior of the fluid. They are called volume or body forces.

A body force is characterized by the force density  $\mathbf{F}(\mathbf{x}, t)$  per unit mass: The force acting on the fluid within a macroscopically small volume  $\delta V$ surrounding point  $\mathbf{x}$  at instant t is

$$\mathbf{F}(\mathbf{x},t)\rho(\mathbf{x},t)\delta V$$

Example: Gravity force:  $\mathbf{g}\rho\delta V$ .

2. Short-range forces (decrease rapidly with distance). They are negligible, unless there is a direct mechanical contact between the interacting fluid elements. (Similar force allows us to sit on a chair).

How to describe such a force? It is determined not only by the point  $\mathbf{x}$  and instant t, but also by the orientation of the surface, passing through that point (i.e. by the unit normal vector  $\mathbf{n}$ ).



The penetration depth is supposed to be small compared with the linear dimensions of the surface element  $\delta A$ . This is possible if the fluid dynamic quantities (like  $\mathbf{u}(\mathbf{x}, t)$ ) change on much larger scale than the penetration depth.

The force per unit area,  $\Sigma(\mathbf{n}, \mathbf{x}, t)$ , is called **stress**. The stress can be represented as the sum of two components: The **normal stress** and the tangential or **shear stress**.

The main property of fluids (which distinguishes a fluid from a solid) is that a fluid cannot withstand a shear stress: Even a slight shear stress makes fluid move.

It interesting to note the following fact.

Melting  $\Rightarrow$  density falls by several percent (water is exception — melting of ice actually increases density). Amazing: Such a small change in the intermolecular spacing leads to such dramatic change in the molecular mobility.

#### 1.2 Hydrostatics — the theory of fluids at rest.

Fluid is at rest.  $\Rightarrow$  There is no shear stress.  $\Rightarrow$  Normal stress is the same in all directions. Show this!

The stress in hydrostatics

$$\Sigma(\mathbf{n}, \mathbf{x}, t) = -p(\mathbf{x}, t)\mathbf{n},$$

where  $p(\mathbf{x}, t)$  does not depend on the orientation of the surface. The function  $p(\mathbf{x}, t)$  is called pressure.

The pressure varies in space.

Example. A static fluid of uniform density in a uniform gravitational field.

$$p = p_0 + \rho gh$$
 Show this!

What is the net force acting on fluid element? The first try. Small cube



The pressure force on the face at x is  $p(x)\delta y\delta z$ . The pressure force on the face at  $x + \delta x$  is  $p(x + \delta x)\delta y\delta z$ . Their resultant is the force in the X direction

$$p(x)\delta y\delta z - p(x+\delta x)\delta y\delta z = -\frac{\partial p}{\partial x}\delta x\delta y\delta z.$$

Similar we find the resultant pressure forces in the other two directions:

Y-direction 
$$-\frac{\partial p}{\partial y}\delta x\delta y\delta z$$
, Z-direction  $-\frac{\partial p}{\partial z}\delta x\delta y\delta z$ 

In the vector form, the net pressure force is

$$-\nabla p\delta x\delta y\delta z.$$

The fluid can also be subject to some long-range forces, with some density (per unit mass)  $\mathbf{F}(x, y, z)$ .

If the force is potential, then  $\mathbf{F} = -\nabla \Phi$ . Example. Uniform gravity  $\Phi = \mathbf{g}z$ .

The long-range force on the cube is

 $\mathbf{F}\rho\delta x\delta y\delta z.$ 

Since the fluid does not move, the sum of these forces should be zero

 $-\nabla p + \rho \mathbf{F} = 0.$ 

This is the equation of hydrostatics.

Although this derivation is quite transparent, it can cause certain dissatisfaction. When we calculated force in the X-direction, we considered variation of pressure along  $\delta x$ , but disregarded the pressure variation along  $\delta y$  and  $\delta z$ . It is not clear *why*. To overcome this shortcoming, we use the divergence theorem of Gauss.

The second try. Volume V of arbitrary shape and size.



The net force acting on this volume is due to (1) the pressure  $p(\mathbf{x})$ , which acts along the boundary S and (2) the long-range force, with some density **F** per unit mass. Since the fluid is at rest, the sum of these two forces equals zero

$$\oint [-p\mathbf{n}] \, dS + \int \mathbf{F} \rho dV = 0.$$

Here and everywhere in the text,  $\mathbf{n} = \mathbf{n}(\mathbf{x})$  denotes the outside normal.

Now we need to apply the Gauss theorem, which states that for any vector field  $\mathbf{h}(\mathbf{x}) = (h_1, h_2, h_3)$ ,

$$\oint \mathbf{h} \cdot \mathbf{n} \, dS = \int \nabla \cdot \mathbf{h} \, dV$$

(the surface S encloses the volume V). With the aid of this theorem we want to transform the surface integral into a volume integral.

Let **a** be an arbitrary constant vector.

$$\mathbf{a} \cdot \oint p\mathbf{n} \, dS = \oint p\mathbf{a} \cdot \mathbf{n} \, dS = \{Gauss\} = \int \nabla \cdot (p\mathbf{a}) dV = \mathbf{a} \cdot \int \nabla p dV.$$

Since **a** is an arbitrary vector (in particular, we could take **a** equal to each of the three basis vectors),

$$\oint p\mathbf{n} \, dS = \int \nabla p dV.$$

Thus,

$$-\int \nabla p dV + \int \mathbf{F} \rho dV = 0.$$

We can make the same argument for any volume V in static fluid; and so,

$$-\nabla p + \rho \mathbf{F} = 0.$$

If the force is potential,

$$-\nabla p - \rho \nabla \Phi = 0.$$

This is the equation of hydrostatics for fluids subject to potential fields. It shows that a fluid can be static only if the density is a function of pressure. You need to show this in the HW Example 1. Hydrostatics of water in a bucket:  $\rho = \text{const}; \ p = p_0 + \rho gh$  Example 2. Atmosphere:  $\rho = \rho(z)$  and p = p(z); so, the density is a function of the pressure.

#### 1.3 Steady flow — Bernoulli's theorem

**Streamlines.** A line whose tangent is everywhere parallel to the velocity **u** is called a **streamline**. If the velocity field  $\mathbf{u} = (u, v, w)$  is known, the family of streamlines at instant t are solutions of the ODE system

$$\frac{dx}{u(\mathbf{x},t)} = \frac{dy}{v(\mathbf{x},t)} = \frac{dz}{w(\mathbf{x},t)}$$

For *steady* flow — when the fluid velocity  $\mathbf{u}$  is time independent, the path of a fluid element coincides with the a streamline. If we observe a tracer particle, following passively the fluid flow, its trajectory is a streamline, provided the flow is steady. If the flow is not steady, the streamlines can be still defined in the same way, but they are not the trajectories of fluid elements.

A **stream-tube** is the surface formed instantaneously by all streamlines that pass through a given closed curve.

#### Assume:

- 1. Fluid is frictionless, and does not conduct heat. So, the energy is conserved.
- 2. Flow is steady (time-independent).
- 3. The long-range force **F** is potential: There is a function  $\Phi(\mathbf{x}, t)$  such that  $\mathbf{F} = -\nabla \Phi$ .

**Then** (Daniel Bernoulli, 1738), the quantity

$$H = \frac{\mathbf{u}^2}{2} + U + \frac{p}{\rho} + \Phi \tag{1}$$

is constant along any streamline (not across streamlines). Here U denotes the internal energy of the fluid per unit mass.

Example. The ideal gas. Its internal energy of a mass M is CMT, where T is the absolute temperature and C is the specific heat (more precisely, specific heat at constant volume,  $C = C_v$ ). So, U = CT.

The fluid between cross-sections 1 and 2 during small time from t to t' has moved along the stream-tube to the volume between the cross-sections 1' and 2'





The change of its energy (it can be calculated considering the end portions only, since the flow is steady)

$$-\left(\frac{\mathbf{u}^{2}}{2}+E+\Psi\right)_{\text{at }\mathbf{x}_{1}} \rho_{1} \,\delta A_{1} \,|\mathbf{u}_{1}|(t^{'}-t) + (...)_{\text{at }\mathbf{x}_{2}} \,\rho_{2} \,\delta A_{2} \,|\mathbf{u}_{2}|(t^{'}-t)$$

is due to the work of the pressure force during time (t' - t)

 $p_1 \,\delta A_1 \,|\mathbf{u}_1| \,(t'-t) \,-\, p_2 \,\delta A_2 \,|\mathbf{u}_2| \,(t'-t).$ 

**Q:** Why isn't there work across the lateral boundary of the stream tube? **A:** There the velocity is orthogonal to the force.

By virtue of the mass conservation,

$$\rho_1 \,\delta A_1 \,|\mathbf{u}_1|(t'-t) = \rho_2 \,\delta A_2 \,|\mathbf{u}_2|(t'-t)|$$

and we establish the constancy of the function (1) along the streamlines.

Bernoulli's theorem in the case of an incompressible fluid. If the fluid is incompressible, and the flow is conservative, the internal energy U stays constant. Indeed, how can the internal energy of a fluid element change? 1. Conduction of heat.

- 2. Internal friction.
- 3. Mechanical work, changing the volume.

The ways 1. and 2. are impossible for conservative flow. The way 3. is impossible if the fluid is incompressible. Then the Bernoulli theorem states that the quantity

$$H = \frac{\mathbf{u}^2}{2} + \frac{p}{\rho} + \Psi$$

is constant along a streamline. Once the density is constant, and  $\Psi$  is known, Bernoulli's theorem provides a simple relation between the fluid speed  $|\mathbf{u}|$  and pressure p.





Demonstration: Ping-Pong ball in an air jet from a leave blower.



Demonstration: Blowing air between two pieces of paper. Try to blow them apart.

# Efflux.



What is the mass flux through an orifice?

In our situation,

$$\frac{p_0}{\rho} = \frac{p_0}{\rho} + \frac{\mathbf{u}_0^2}{2} - gh \quad \Rightarrow \quad |\mathbf{u}_0| = \sqrt{gh}.$$

(This is the speed attained in free fall from height h).

It is tempting to say that the efflux (fluid mass flowing out per second) is

$$M = \rho S |\mathbf{u}_0|,$$

where S is the area of the orifice,  $\rho$  is the density of the fluid.

However, this is not what observed in experiments.

The streamline of the jet are converging for a few diameters after the orifice. The cross-sectional area of the jet — when the streamlines become parallel — is smaller by some factor  $\alpha$  than the area S of the orifice, and so the efflux

$$M = \rho(\alpha S) |\mathbf{u}_0|.$$

The factors  $\alpha$  depends on the shapes of discharge holes; these factors are available in tables of *efflux coefficients*.

In the case of a circular hole in a thin wall, the experiments give 0.61  $< \alpha < 0.64.$ 

If there is a re-entrant discharge tube,



then it is possible to find that  $\alpha$  is exactly  $\frac{1}{2}$ . The derivation of this fact uses conservation of momentum in addition to Bernoulli's theorem (which is the conservation of energy).

Each second, the fluid acquires momentum  $M|\mathbf{u}_0|$ . Why?

Because of to the force from the vessel  $= \rho ghS$ . (Near the walls the fluid is almost at rest, and the pressure is hydrostatic  $p = \rho gh$ . The force from the 'left' wall is balanced by the force on the 'right' wall everywhere, except from the piece of wall opposite to the hole.)

Thus,  $\rho \alpha S |\mathbf{u}_0|^2 = \rho g h S$ . Together with the Bernoulli's expression for the speed  $|\mathbf{u}_0|$ , this gives  $\alpha = 1/2$ .

#### Steady flow of a perfect gas.



Neglect the body forces.

The internal energy (per unit mass)  $E = C_v T$ . According to the equation of state,  $\frac{p}{\rho} = \frac{RT}{m}$  where R is the universal gas constant, m is the mole mass. Then the Bernoulli theorem

$$(C_v + \frac{R}{m})T + \frac{\mathbf{u}^2}{2} = \text{const}$$

gives a simple relation between the speed  $|\mathbf{u}|$  and the temperature T along a streamline. The gas is hotter at places on the streamline where the speed is smaller.

Recall: If we neglect body forces in an incompressible flow, we find that along a streamline the pressure is bigger when the speed is smaller:

$$\frac{\mathbf{u}^2}{2} + \frac{p}{\rho} = \text{constant along a streamline.}$$

#### 1.4 The equations of Fluid Mechanics

**Mass conservation** — continuity equation Consider the volume V whose position is fixed relative to the coordinate axis (not moving with the fluid). The total mass of the fluid in the volume V is

$$\int \rho dV.$$

In the absence of sources and sinks of the fluid, this mass changes only due to the net flux of the fluid through the boundary. How much fluid mass does go through the boundary?

$$-\oint \rho \mathbf{u} \cdot \mathbf{n} dA.$$

Thus,

$$\frac{d}{dt}\int\rho dV = -\oint\rho\mathbf{u}\cdot\mathbf{n}dA$$

On the left, differentiate in t under the integral sign (remembering that the volume V is fixed). On the right transform the surface integral to the volume integral using the Gauss theorem. Then

$$\int \left\{ \frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{u}) \right\} dV.$$

Since the volume V is arbitrary,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{2}$$

Momentum conservation — Newton's law Our goal is to write Newton's second law

$$mass \times acceleration = force$$

for fluid motion.

**Mass.** Consider a "small" fluid element with volume  $\delta V$  and density  $\rho$ . Its mass is  $\rho(\mathbf{x}, t)\delta V$ .

Acceleration. It is clear that  $\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t}$  is not an acceleration of a fluid particle: Fluid flow can be stationary  $(\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t} = 0)$ , but the forces are present. For instance, consider fluid in solid body rotation.

To find the correct expression for the acceleration of a fluid particle, we should note that a fluid particle at position  $\mathbf{x}$  at instant t will be at instant  $t + \delta t$  at position

$$\mathbf{x} + \mathbf{u}\delta t$$
 where  $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ 



The change in its velocity  $u_i$  (i = 1, 2, 3)

$$u_i(\mathbf{x} + \mathbf{u}\delta t, t + \delta t) - u_i(\mathbf{x}, t) = \mathbf{u}\delta t \cdot \nabla u_i + \delta t \frac{\partial \mathbf{u}}{\partial t} + O(\delta t)^2.$$

So the *i*-th component of acceleration is

$$\frac{\partial u_i}{\partial t} + \mathbf{u} \cdot \nabla u_i$$
 which we denote by  $\frac{Du_i}{Dt}$ .

The operator  $\frac{D}{Dt}$  gives time derivative following the motion of the fluid, or a material derivative; it has meaning only when applied to a field variable (which depends on **x**, besides t).

In vector form the fluid acceleration is

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}.$$

**Force.** The total force is the sum of the long-range force and the short-range force. Since for now we neglect viscosity (or any friction), the short-range force is due to pressure.

The long-range force is

$$(\rho(\mathbf{x},t)\delta V)\mathbf{F}(\mathbf{x},t).$$

The pressure  $p(\mathbf{x}, t)$  creates a force on this fluid element (as we know from hydrostatics) equal to

 $-\nabla p \delta V.$ 

The equation of Euler (1707-1783). Combining mass, acceleration, and force into Newton's second law, we find

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{F}(\mathbf{x}, t), \tag{3}$$

which is called Euler's equation (1755).

The fluid dynamics system The continuity equation (2) and Euler's equation (3) compose a system of 4 equations for 5 unknown functions — the 3 velocity components  $\mathbf{u}(\mathbf{x},t) = (u, v, w)$ , the pressure  $p(\mathbf{x},t)$ , and the density  $\rho(\mathbf{x},t)$ . So, we need one more equation. The fifth partial differential equation is the equation of the energy conservation. However, it involves the temperature  $T(\mathbf{x},t)$ , and now we need the sixth equation. This is the equation of state, that connects p,  $\rho$ , and T; solving it for  $\rho$ , we would find the fluid density as a given function of the fluid pressure and temperature:  $\rho = \rho(p, T)$ . A fluid is called *barotropic* if its density  $\rho$ , is a function of the pressure p.

A fluid is called *ideal* if (1) it is incompressible and its density is constant throughout the fluid and (2) the short-range is due to the pressure. An ideal fluid is trivially barotropic.

The flow of an ideal fluid is described by the system

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla \left(\frac{p}{\rho}\right) + \mathbf{F}(\mathbf{x}, t), \tag{4}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{5}$$

#### 1.5 The speed of sound

With which speed does our sound propagate?

The sound is a wave of density and pressure variations in space and in time. So, we cannot assume that the density is constant throughout the fluid and cannot use the system of equations for an ideal fluid.

Take the system of Euler's equation and the mass-conservation equation. Neglect gravity forces.

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p,$$
  
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

Exact solution:  $\mathbf{u} \equiv 0$ ,  $\rho(\mathbf{x}, t) \equiv \rho_0 = const$ ,  $p(\mathbf{x}, t) \equiv p_0 = const$ . This is an equilibrium (still air).

Now, let us linearize our equations around this exact solution: Take

$$\rho = \rho_0 + \tilde{\rho}, \quad p = p_0 + \tilde{p}, \quad \mathbf{u} = 0 + \tilde{\mathbf{u}},$$

and neglect products of small quantities  $\tilde{\rho}$ ,  $\tilde{p}$ ,  $\tilde{\mathbf{u}}$ .

$$\rho_0 \frac{\partial \tilde{\mathbf{u}}}{\partial t} = -\nabla \tilde{p},$$
$$\frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} = 0.$$

Why can we linearize? The pressure fluctuations in a sound traveling in air are typically  $10^{-9}$  to  $10^{-5}$  of the atmospheric pressure. It makes no sense to keep products of so small quantities compared to these quantities themselves. We can easily exclude the velocity  $\tilde{\mathbf{u}}(\mathbf{x}, t)$ 

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} = -\nabla \cdot \left( \rho_0 \frac{\partial \tilde{\mathbf{u}}}{\partial t} \right) = \nabla^2 \tilde{p}.$$

Already Newton did this calculation. To calculate the speed of sound, he needed the relation between the pressure and the density (then he would have 1 equation with 1 unknown). Some years before Newton, Boyle did such experiments



He found that

$$\frac{p}{\rho} = \frac{p_0}{\rho_0} = \text{const}$$
 (Boyle's law).

Then our equation becomes the wave equation

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} = c^2 \nabla^2 \tilde{\rho}$$

with the speed of sound  $c = \sqrt{p_0/\rho_0}$ .

For normal air (at 15 degrees C),

$$p_0 = 1$$
atm  $= 10^5 \frac{kg}{m \cdot s^2}, \quad \rho_0 = 1.2 \frac{kg}{m^3} \quad \Rightarrow \quad c = 290 m/s.$ 

You can imagine how greatly Newton was disappointed when he compared his theoretical value with the experimental value c = 340m/s.

What is wrong? Only 100 years later, Laplace explained this disagreement.

In the Boyle experiment, the air is compressed slowly, so that its temperature equilibrates to the room temperature; so, the air is compressed at the constant temperature, i.e. isothermally. In the sound wave the air is compressed adiabatically, i.e. without heat transmission; the compression is so fast that the fluid elements have no time to exchange heat.

This idea leads to a different equation of state

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \tag{6}$$

where  $\gamma$  is a constant exponent; for air,  $\gamma \approx 1.4$  (before we had the equation (6) with  $\gamma = 1$ ). First, let us see how the new equation of state arises.

First law of thermodynamics. The internal energy (U) of a of a fluid is changed by a gain of heat (Q) and by the performance of work (W):

$$\Delta U = Q + W$$



The internal energy is a function of the parameters of the system, and so,  $\Delta U$ depends only on the initial and final states. On the contrary, Q and W depend on the *way* of transition between the initial and final states.

For adiabatic process (without heat exchange), Q = 0, the internal energy (U) is changed only by the performance of work (W).

Suppose, the fluid is compressed slowly enough for the pressure p to equilibrate to a uniform pressure throughout the gas element; then W = -pdV, where dV is the volume change.

The energy of an ideal gas is  $E = MC_v T$  where M is the mass of the gas element, T is the temperature of the gas, and  $C_v$  is the specific heat ( $C_v$  shows how much heat we need to transfer to 1 kilogram of the gas to increase its temperature by 1 degree K, if the volume of the gas is held fixed).

Thus, the first law of thermodynamics gives

$$MC_v dT = -pdV.$$

The equation of state for an ideal gas. The first law of thermodynamics gives us one equation relating three quantities temperature T, pressure p and volume V of a fluid element with a constant mass M. We need one more equation to find the relation between p and  $\rho$ . This will be the equation of state

$$pV = \frac{M}{m}RT$$

where R is the universal gas constant, m is the mass of one mole of the gas.

**Q:** What is a mole?

**A:** A mole of something is the Avogadro number  $N_A ~(\approx 6 \cdot 10^{23})$  of these things.

 $N_A$  shows how many protons (or neutrons) we need to get 1 gram. More precisely,  $N_A$  is defined as the number atoms in 12 grams of carbon  $C^{12}$ . (Neutrons and protons have almost the same mass, and the electrons are almost mass-less.)

**Q:** How much does a mole of hydrogen  $H_2$  weigh?

A: 2 grams.

 $\frac{M}{m}$  is the number of moles in the gas element of mass M.  $\frac{M}{m}N_A$  is the number of molecules in the gas element of mass M.

The equation of the ideal gas can be written in the form

$$pV = \frac{M}{m}N_A \cdot k_B T$$
 (k<sub>B</sub> is the Boltzmann constant).

It coincides with the previous form once  $R = N_A k_B \ (R \approx 8.3 \frac{J}{mol \cdot K})$ .

**True speed of sound.** From the equation of state, we can express the temperature T in terms of p and V. Substituting it into the first law of thermodynamics, we find

$$\frac{dp}{p} = -\gamma \frac{dV}{V}$$
 where  $\gamma = 1 + \frac{R}{mC_v}$ 

Instead of the volume V we can use the density  $\rho = M/V$ 

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho}$$

Integrating this differential equation, we find the equation of state (6).

By Taylor's formula expansion,

$$p - p_0 = \left. \frac{dp}{d\rho} \right|_{\text{at } \rho_0} (\rho - \rho_0), \quad \Rightarrow \quad \tilde{p} = \gamma \frac{p_0}{\rho_0} \tilde{\rho}$$

We again have the wave equation, but with a different speed

$$c = \sqrt{\gamma \frac{p_0}{\rho_0}}.$$

For the air,  $\gamma = 7/5$ , and the sound speed is c = 340m/s, which agrees with experiments. The new value is bigger than Newton's one by the factor  $\sqrt{\gamma}$ .

**Fast or Slow?** We said that in the sound wave the compression is so fast that the fluid elements have no time to exchange heat.

We also said that the compression is slow enough that the pressure is in time to equilibrate to a uniform thermodynamic pressure.

How can it be that the compression is fast in one respect and slow in another? Answer: The pressure equilibrates much faster (with the sound speed) than temperature (that equilibrates diffusively).

#### **1.6** Stability of the atmosphere

How much the air temperature can vary with height if the atmosphere is stable?



Equilibrium: Any  $\rho = \rho(z)$ . Example: Water (of 1 m depth) on the ceiling.

If the fluid were incompressible... ... The atmosphere would be stable if  $\rho(z)$  were a decreasing function. Why?

If a fluid element is displaced upward, it would find itself among lighter fluid, and it would be forced back.

In the compressible case, the decrease of the density  $\rho(z)$  is not enough, the density should not be just decreasing function of z, but it should decrease sufficiently fast.

Indeed, when a fluid element at a level z, with density  $\rho(z)$ , is displaced upward to level z + dz, it would feel less pressure, and therefore it would expand, and its density would decrease to  $\rho_*$  (which is smaller than  $\rho(z)$ ). The atmosphere is stable if the displaced fluid element is forced back, i.e. the new density  $\rho_*$  exceeds the surrounding density  $\rho(z + dz)$ .



The temperature T also changes with height.  $\Rightarrow$  The displaced fluid element would find itself surrounded by gas at a different temperature.  $\Rightarrow$  Heat exchange would occur. However, it is a slow process. We can assume that the fluid element is displaced adiabatically

$$\frac{p - \rho g dz}{p} = \left(\frac{\rho_*}{\rho}\right)^{\gamma} \quad \Rightarrow \quad \rho_* = \rho - \frac{\rho^2 g}{\gamma p} dz.$$

We should compare this density  $\rho_*$  with the surrounding density  $\rho + \rho' dz$ . Thus, the condition for the stability of the atmosphere is

$$\rho' < -\frac{\rho^2 g}{\gamma p}.$$

The density of the atmosphere should not just decrease with height, but decrease sufficiently fast.

**The temperature variation.** How does the temperature need to vary with height in order for the atmosphere to be stable? We will answer this question assuming that the atmosphere consists of a perfect gas

$$\frac{p}{\rho} = \frac{R}{m}T \quad \Rightarrow \quad T' = \frac{m}{R}\frac{p'\rho - p\rho'}{\rho^2}.$$

Using the hydrostatic balance,  $p' = -\rho g$ , and the stability condition in terms of the density  $\rho$ , we find

$$T' > \frac{m}{R}g(-1 + \frac{1}{\gamma}).$$

For the earth atmosphere:

$$\gamma = 1.4, \ m = 0.029 kg/mol, \ R = 8.3 \frac{J}{mol \cdot K}, \ g = 9.8 m/s^2 \quad \Rightarrow \quad T' > 0.01 \frac{K}{m}.$$

So, the temperature can decrease with height at most 10 degrees (K or C) per kilometer. The lower atmosphere turns out to be close to being neutrally stable, and so, this rate is often observed.

#### 1.7 Vorticity

The vorticity is defined as curl of the velocity:

$$\mathbf{\Omega} = \nabla \times \mathbf{u}.$$

**Euler's equation in terms of the vorticity.** We can rearrange Euler's equation by using the following identity from vector analysis

$$\frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) = \mathbf{u}\times(\nabla\times\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u}.$$

So, Euler's equation becomes

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{u}) = -\frac{1}{2}\nabla(u^2) - \frac{1}{\rho}\nabla p - \nabla\Phi.$$

Here we assumed that the long-range force is potential,  $\mathbf{F} = -\nabla \Phi$ .

We can eliminate the pressure from our equations. If the density is constant throughout the fluid, then the right-hand side is the gradient of some function,

$$\nabla\left(-\frac{u^2}{2} - \frac{p}{\rho} - \Phi\right)$$

Taking the curl of both sides of our equation, we find

$$\frac{\partial \nabla \times \mathbf{u}}{\partial t} - \nabla \times [\mathbf{u} \times (\nabla \times \mathbf{u})] = 0.$$

The vorticity has the principal importance in fluid dynamics. In terms of the vorticity, Euler's equation takes the form

$$\frac{\partial\Omega}{\partial t} + \nabla \times (\Omega \times \mathbf{u}) = 0.$$

This equation together with the equations

$$\nabla \times \mathbf{u} = \Omega$$
 and  $\nabla \cdot \mathbf{u} = 0$ 

describes completely the velocity field **u**. Indeed, if we know  $\Omega$  at some instant, then at that instant we know the curl and the divergence of the velocity field. This is enough to find the velocity field itself. Then we know the rate of change of the vorticity field, and we can get  $\Omega$  for the next instant.

We can eliminate the pressure for any barotropic flow (it is not necessary to require that the density is constant throughout the fluid). Indeed, when the density is a function of pressure, we still have that  $(1/\rho)\nabla p$  is a gradient of some function:

$$\frac{1}{\rho(p)}\nabla p = \nabla G, \quad \text{where } G = G(p) \text{ is an anti-derivative of } \frac{1}{\rho(p)} \\ \left(\nabla G = \frac{dG}{dp}\nabla p = \frac{1}{\rho(p)}\nabla p\right).$$

The vorticity is a double angular velocity. Consider a "small" circle A around point  $\mathbf{x}$ ; radius a; boundary l



**Q:** What would you call the angular velocity  $\omega$  of fluid rotation at point **x**? **A:**  $\omega \cdot \mathbf{n}$  should equal the average velocity component tangential to the boundary, divided by the radius *a*:

$$\omega \cdot \mathbf{n} = \frac{1}{a} \left( \frac{1}{2\pi a} \oint \mathbf{u} \cdot \mathbf{dl} \right).$$

Applying the Stokes theorem

$$\oint \mathbf{u} \cdot \mathbf{dl} = \int (\nabla \times \mathbf{u}) \cdot \mathbf{n} dA,$$

we find

$$\omega \cdot \mathbf{n} \approx \frac{1}{a} \left( \frac{1}{2\pi a} \nabla \times \mathbf{u} \cdot \mathbf{n} |_{at\mathbf{x}} \pi a^2 \right) = \frac{1}{2} \Omega \cdot \mathbf{n} |_{at\mathbf{x}}$$

A bucket on a turntable. Consider a bucket of water rotating on a turntable with angular velocity  $\omega$ . Let us calculate the vorticity in the water.

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} u = -\omega y \\ v = \omega x \\ w = 0 \end{bmatrix} \longrightarrow \Omega = \begin{bmatrix} 0 \\ 0 \\ 2\omega \end{bmatrix}.$$

A shear flow.

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \beta y \\ 0 \\ 0 \end{bmatrix} \longrightarrow \Omega = \begin{bmatrix} 0 \\ 0 \\ \beta \end{bmatrix}.$$

An irrotational flow. A flow is called *irrotational* if the vorticity is zero everywhere:  $\Omega \equiv 0$ .

If  $\Omega \equiv 0$  at some instant t, then — according to our equation —  $\partial\Omega/\partial t \equiv 0$ , so that  $\Omega$  is still zero everywhere at t + dt. Thus if the flow is irrotational initially, it will remain irrotational for ever.

# 2 Waves

## 2.1 Introduction

People are often fascinated when observing water waves (on the surface of an ocean, lake, water tank, or plate). It is so easy to make these observations. Yet, they naturally rise many important questions, which are not answered for tens, hundreds, even thousands of years.

### Example. Calming rough water with oil.

It was noted at least couple thousands of years ago that spilling oil on the surface of an agitated sea will significantly reduce the waves.

Even small amount of oil is sufficient.

Benjamin Franklin in a letter dated 1773 describes the following examples:

- Fishery boats in Portugal when returning to the harbors, if the fishermen see quite rough sea, they pour out a bottle of vegetable oil to ensure safe passage.
- In 1757 Franklin traveled in a fleet of 96 sail ships. He noticed that the water in the wake of two ships was "remarkably smooth", while the wakes behind all other ships was rough as everywhere in the sea. He asked captain about this strange phenomenon. "The cooks," answered the captain. They poured out their greasy water, which calmed the sea surface in the wake.
- In Mediterranean, the divers take some oil in their mouths. If there are many waves on the surface which irregularly reflect sunlight, the divers release small quantities of oil from their mouths.
- Franklin, himself made an experiment at a large pond. He was able to calm half an acre of rough water surface with a teaspoon of oil. He also describes many other situations.

This phenomenon had been used for life saving in the sea. For example, in UK the saving services had some special surfactants in order to calm the sea and facilitate the life saving.

Despite significant efforts, this phenomenon still remains unexplained. Many qualitative mechanisms for this phenomenon were suggested, but they all turned to be insufficient.

### 2.2 The dynamics of free surface

The surface is moving, and we should determine its shape together with the velocity field.

**Frame:** axis y is vertical, in the opposite direction to the gravity acceleration **g**; plane (x, z) is horizontal, orthogonal to the gravity acceleration **g**.



Assume:

- The gravity field is constant
- No solid bodies in the fluid
- The fluid is ideal (it is inviscid, and its density is constant)
- The flow is irrotational

Why can we make the last assumption? Remember the last sentence in the previous chapter: If the flow is irrotational initially, it will remain irrotational for ever. So, we just assume that the flow is irrotational initially.

Recall the following fact from the vector analysis. Suppose a vector field  $\mathbf{u}(\mathbf{x})$  is defined in a simply connected domain D. (What does it mean "simply connected"?) If  $\nabla \times \mathbf{u} \equiv 0$  in D, then there is a function  $\phi(\mathbf{x})$  such that  $\mathbf{u} = \nabla \phi$ .

How do you see this? Fix any point  $\mathbf{x}_0$  in D. Take any constant as  $\phi(\mathbf{x}_0)$ . Then define

$$\phi(\mathbf{x}) = \phi(\mathbf{x}_0) + \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{u} \cdot \mathbf{dl},$$

where l is a curve going from the point  $\mathbf{x}_0$  to the point  $\mathbf{x}$ . We need to make sure that this definition is consistent. Namely, the value of the function  $\phi(\mathbf{x})$ is independent of the curve l joining the points  $\mathbf{x}_0$  and  $\mathbf{x}$ .



Indeed, consider a counter clock-wise closed contour C consisting of  $l_2$  and  $l_1$  taken in the opposite direction (from  $\mathbf{x}$  to  $\mathbf{x}_0$ )

$$\int_{l_2} \mathbf{u} \cdot \mathbf{dl} - \int_{l_1} \mathbf{u} \cdot \mathbf{dl} = \int_C \mathbf{u} \cdot \mathbf{dl} = \langle \text{ Stokes theorem } \rangle = \int_A \nabla \times \mathbf{u} dA = 0$$

(the surface A entirely lies in the domain D).

Applying this fact to fluids, we note: (1) The domain filled with the fluid is simply connected, since there are no solid bodies in the fluid. (2) The flow is irrotational. Hence the velocity field is potential: There is a function  $\phi(\mathbf{x}, t)$ , such that  $\mathbf{u} = \nabla \phi$ .

Since the fluid is incompressible,  $\nabla \cdot \mathbf{u} = 0$ , the function  $\phi(\mathbf{x}, t)$  should satisfy the Laplace equation

$$\Delta \phi = 0. \tag{7}$$

**Q:** Does it mean there is no dynamics (see, the equation is static) ?

A: The dynamics comes from the boundary conditions. The moving boundary  $y = \eta(x, z, t)$  completely determines the velocity field inside the fluid.

We also see that the fluid velocity is determined by a linear equation! The nonlinear Euler equation becomes only equation for the calculation of pressure. Using the identity

$$\frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) = \mathbf{u}\times(\nabla\times\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u}.$$

we re-write Euler's equation in the form

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{u}) = -\frac{1}{2}\nabla(u^2) - \frac{1}{\rho}\nabla p - \nabla\Phi.$$

(Here we assumed that the long-range force is potential,  $\mathbf{F} = -\nabla \Phi$ .) When the density is constant, and the flow is irrotational,  $\mathbf{u} = \nabla \phi$ , this equation becomes

$$\nabla \cdot \left(\frac{\partial \phi}{\partial t} + \frac{u^2}{2} + \frac{p}{\rho} + \Phi\right) = 0.$$

This means that

 $\frac{\partial \phi}{\partial t} + \frac{\mathbf{u}^2}{2} + \frac{p}{\rho} + \Psi(\mathbf{x}) = C \quad \text{(const throughout the fluid)}.$ 

It is almost like Bernoulli's theorem for steady fluid, except that now the constant, has the same value for different streamlines, and the flow is not necessarily steady. The price that we paid for such generalization: The flow has to be irrotational.

The constant C can be a function of time t. However, it can be assumed equal zero, or any other constant, since C(t) can be included in the potential  $(\phi \rightarrow \phi + \int C(t)dt$ ; this does not change the velocity field). Later we chose Cequal to the atmospheric pressure  $p_0$  divided by the density  $\rho$ ,  $C = p_0/\rho$ .

Thus, solving the Laplace equation (7), we find the velocity field  $\mathbf{u} = \nabla \phi$ and the pressure throughout the fluid by the unsteady Bernoulli equation.

The nonlinearity of free surface dynamics comes from the boundary conditions at the free surface.

Let the free surface have the equation

$$y = \eta(x, z, t).$$

**Dynamic boundary condition.** The pressure in the fluid at the free surface satisfies the equation

$$\frac{\partial \phi}{\partial t} + \frac{\mathbf{u}^2}{2} + \frac{p}{\rho} + g\eta = C \quad \text{(const throughout the fluid)}.$$

Since the free surface has no mass, the forces at the surface from the water side and from the air side must be equal. In the absence of viscosity these forces are due to the pressure.

For now, we disregard surface tension. Then the pressure in the water and the pressure in the air must be equal at the surface. We assume, at present, that the air pressure is constant  $p_0$ . This is because the density of the air is so much smaller than the density of the water, and the pressure variations — according to the Bernoulli theorem — are of the order density(velocity<sup>2</sup> +  $g \times \text{hight}$ ). Later, we will consider the dynamics of an interface between two fluids with different densities.

Thus, at the free surface, the pressure p in the water should be equal to the constant pressure  $p_0$  in the air

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + g\eta = 0$$
 at the free surface  $y = \eta(x, z, t)$ .

This is called the *dynamic boundary condition*.

**Kinematic boundary condition. Q:** Should the velocity of the surface coincide with the velocity of the fluid at the surface?

A: No. The fluid can slide along the surface. Only their normal velocities should coincide. "Normal" means "normal to the surface at the location of the fluid element. The normal velocity of the fluid at the surface should be equal to the normal velocity of the surface. This implies that if a fluid particle is at the surface at some instant, it will remain at the surface for ever. If at instant t a fluid particle is at location (x, y, z), then at instant t + dt it is at location (x + u dt, y + v dt, z + w dt). The particle is at the surface if

at instant 
$$t: \quad y = \eta(x, z, t)$$
  
at instant  $t + dt: \quad y + v \, dt = \eta(x + u \, dt, z + w \, dt, t + dt)$   
 $\Rightarrow \quad v = \frac{\partial \eta}{\partial x} u + \frac{\partial \eta}{\partial z} w + \frac{\partial \eta}{\partial t}.$ 

The later equation in terms of the potential is

 $\phi_y = \eta_t + \phi_x \eta_x + \phi_z \eta_z$  at the free surface  $y = \eta(x, z, t)$ ,

which is called the *kinematic boundary condition*.

**Boundary conditions at the bottom.** We also should have boundary condition at the bottom, which is a rigid boundary. At first, we assume that the water is infinitely deep, so that

$$\frac{\partial \phi}{\partial y} \to 0$$
 as  $y \to -\infty$ .

The other fluid velocities

$$u = \frac{\partial \phi}{\partial x}$$
 and  $w = \frac{\partial \phi}{\partial z}$ 

do not need to vanish when  $y \to -\infty$ . It depends on the frame of reference moving horizontally. (In any such frame the fluid domain is not changing.) They do vanish only in the frame fixed at the bottom.

We have one boundary condition at the rigid bottom but two boundary conditions at the free surface. This is because the free surface itself should be determined along with the potential  $\phi$ .

**System.** Thus the free surface dynamics is determined by the following system

 $\begin{array}{lll} \mbox{Laplace eq.} & \Delta\phi=0 & \mbox{in the fluid domain } y<\eta(x,z,t), \\ \mbox{Dynamic b.c.} & \phi_t+\frac{1}{2}(\nabla\phi)^2+g\eta=0 & \mbox{at the free surface } y=\eta(x,z,t), \\ \mbox{Kinematic b.c.} & \phi_y=\eta_t+\phi_x\,\eta_x+\phi_z\,\eta_z & \mbox{at the free surface } y=\eta(x,z,t), \\ \mbox{Bottom b.c.} & \frac{\partial\phi}{\partial y}\to 0 & \mbox{as } y\to-\infty. \end{array}$ 

#### 2.3 Linear surface gravity waves

Exact solution: the rest state

$$\eta(x, z, t) \equiv 0, \quad \phi(x, y, z, t) \equiv 0.$$

Let us consider small perturbations on the background of this solution. For this we linearize our equations, neglecting the products of small quantities. First, we neglect  $(\nabla \phi)^2$ , as well as  $\phi_x \eta_x$  and  $\phi_z \eta_z$ . Second, using Taylor's expansion, we replace the values of  $\phi_t$ ,  $\phi_y$ , and  $\eta_t$  at the fluid surface  $y = \eta$ by their values at the plain y = 0 (because  $\eta$  is small).

 $\begin{array}{lll} \Delta \phi = 0 & \qquad \text{in the half space } -\infty < y < 0, \\ \phi_t + g\eta = 0 & \qquad \text{when } y = 0, \\ \phi_y = \eta_t & \qquad \text{when } y = 0, \\ \frac{\partial \phi}{\partial y} \to 0 & \qquad \text{as } y \to -\infty. \end{array}$ 

We can solve this linear boundary value problem by the method of separation of variables. Since the problem is invariant with respect to translations in x, in z, and in t, we already know that the dependence on x, z, and t is in the form of exponents

$$\eta = Ae^{i(px+qz-\omega t)}, \quad \phi = \Phi(y)e^{i(px+qz-\omega t)}.$$

Since, the solution should not grow at infinity,  $(x, z) \to \infty$ , the exponents p and q should be real. We expect oscillations in time; this would be indeed the case if  $\omega$  turned out to be real as well.

The linearized fluid equations give

$$\begin{split} \Phi'' &= k^2 \Phi, & -\infty < y < 0 \quad (k = \sqrt{p^2 + q^2}), \\ -i\omega \Phi(0) &= -gA, \\ \Phi'(0) &= -i\omega A, \\ \Phi'(-\infty) &= 0. \end{split}$$

From the first, the second, and the fourth condition we find

$$\Phi = \frac{gA}{i\omega}e^{ky}$$

In order to satisfy the third condition, we need to require

$$\omega^2 = gk \quad \Rightarrow \quad \omega = \pm \Omega_{\mathbf{k}} \qquad \text{where} \quad \Omega_{\mathbf{k}} = \sqrt{gk}.$$

The function  $\Omega_{\mathbf{k}}$  is the **dispersion law** of surface gravity waves over deep water. The constant A remains arbitrary; it has the meaning of the wave amplitude. We see that the frequency  $\omega$  turned out to be *real*, and we have found waves (not decaying and not growing).

#### 2.4 Dispersion law: General theory

The dispersion law arises in many situations, and it has principal importance. Let us consider several simple examples.

• Linear Kortveg-deVries equation

$$\frac{\partial \phi}{\partial t} + \alpha \frac{\partial \phi}{\partial x} = \beta \frac{\partial^3 \phi}{\partial x^3} \qquad (-\infty < x < \infty).$$

How to solve this equation?

We can find solutions by the method of separation of variables. Since the equation is invariant with respect to translations in x and in t, we already know the form of the solutions

$$\psi = Ae^{i(kx - \omega t)},$$

where  $k, \omega$ , and A are constants.

Let us see this. Try  $\psi(x,t) = X(x)T(t)$ 

$$XT' + \alpha X'T = \beta X'''T \qquad \Rightarrow \qquad \frac{T'}{T} = \frac{-\alpha X' + \beta X'''}{X} = C = \text{Const.}$$

The general solution of this ODE is a linear combination of exponential functions. Since the solution should be bounded on the whole line  $-\infty < x < \infty$ , the exponential solutions should have the imaginary exponents

$$X = e^{ikx}$$
 (k is a real constant).

Hence

$$C = -i\beta k^3 - i\alpha k$$
, and  $T = e^{-i\Omega_k t}$ , where  $\Omega_k = \alpha k + \beta k^3$ 

So, using the method of separation of variables, we have found the solutions

$$e^{i(kx-\Omega_k t)}$$
, where  $\omega = \Omega_k$ .

The general solution is a superposition of those solutions

$$\psi(x,t) = \int A_k e^{i(kx - \Omega_k t)} dk.$$

The function  $\Omega_k$  is called the dispersion law of the linear Kortveg-deVries equation.

• Linear Klein-Gordon equation

$$\frac{\partial^2 \psi}{\partial t^2} - \alpha^2 \frac{\partial^2 \psi}{\partial x^2} + \beta^2 \psi = 0, \qquad (-\infty < x < \infty).$$

We can find solutions by the method of separation of variables. Since the equation is invariant with respect to translations in x and in t, we already know the form of the solutions

$$\psi = A e^{i(kx - \omega t)},$$

where  $k, \omega$ , and A (the amplitude) are constants. Substituting this function into our equation, we find that it is indeed a solution provided

$$\omega = \pm \Omega_k$$
, where  $\Omega_k = \sqrt{\alpha^2 k^2 + \beta^2}$ .

The function  $\Omega_k$  is called the dispersion law of the linear Klein-Gordon equation. The general solution is a superposition of these simple solutions

$$\psi(x,t) = \int A_k e^{i(kx - \Omega_k t)} dk + \int B_k e^{i(kx + \Omega_k t)} dk$$

• Wave equation. When  $\beta = 0$ , the linear Klein-Gordon equation becomes the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = \alpha^2 \frac{\partial^2 \psi}{\partial x^2}, \qquad (-\infty < x < \infty).$$

Its solutions are linear combinations of exponents

$$\psi = Ae^{i(kx - \omega t)},$$

where

$$\omega = \pm \Omega_k$$
 with the dispersion law  $\Omega_k = \alpha k$ .

Harmonic wave. The exponent

$$\psi = Ae^{i(kx - \omega t)}$$

is called harmonic wave. Its real part is usual sinusoid

$$a\cos(kx - \omega t - \theta)$$
 where  $A = ae^{i\theta}$ .

It represents harmonic oscillations in time and in space.

What is the period of these oscillations in time?

$$\tau = \frac{2\pi}{\omega};$$
  $\omega$  is called the wave frequency.

What is the period of these oscillations in space? In other words, after which length does the solution repeat itself?

 $\lambda = \frac{2\pi}{k}$ ;  $\lambda$  is called the wave length; k is called the wave number.

What is the speed of a harmonic wave?

$$c = \frac{\Omega_k}{k} = \frac{\lambda}{\tau}.$$

The maxima of a wave are called *crests*. The minima of a wave are called *troughs*.

**Dispersion** It often happens that different waves (waves with different wave vectors) propagate with different velocities. This is called *dispersion*; and such waves are called *dispersive*.

In the above examples, the waves are dispersive in the cases of the linear Kortveg-deVries equation and linear Klein-Gordon equation; but they are not dispersive in the case of the Wave equation. Indeed, for the wave equation, the wave speed is constant

 $c = \alpha$  for all wave numbers.

Are the surface gravity waves dispersive? Their speed

$$c = \frac{\Omega_k}{k} = \sqrt{\frac{g}{k}}$$

depends on the wave number k.

Typical wave length of a surface gravity wave in the ocean ranges from about 0.1m to 100m and even longer.

$$\lambda = 0.1m \rightarrow T = 0.25s \text{ and } c = 0.4m/s,$$
  
 $\lambda = 100m \rightarrow T = 8s \text{ and } c = 13m/s.$ 

We see significant dispersion: The wave speed varies substantially over the range of wavelengths of interest.

Why are waves important? The importance of waves stems from the fact that they transport the energy and information over long distances almost without changing the medium. Recall electromagnetic waves, sound waves, and sea waves. Without waves we would not see or hear anything.

However, the speed

$$c = \frac{\Omega_k}{k} = \frac{\lambda}{\tau}$$

is not the speed with which the energy and information are transported. This speed is called *phase speed*. Waves transport the energy and information with the **group velocity** 

$$C = \frac{d\Omega}{dk} \; .$$

On the graph of the dispersion law  $\Omega_k$  vs. k we can easily see the group and phase velocities. Consider, for example, the linear Klein-Gordon equation



For the waves with the wave number  $k_0$ , the group velocity C is the slope of the tangent to curve  $\omega = \Omega_k$  at  $k_0$ ; the phase velocity c is the slope of the straight line passing through the origin and the point on the graph with abscissa  $k_0$ .

We see that

as 
$$k_0 \to 0$$
:  $C(k_0) \to 0$ ,  $c(k_0) \to \infty$ .

So, the group and the phase velocities can be quite different.

#### 2.5 Group velocity: Propagation of Fourier harmonics

The notion of group velocity is very important, and its understanding is not simple. How can a wave travel with velocity other than  $c = \lambda/\tau$ ? We will consider the group velocity in five ways, each revealing certain prop-

erties of this notion. **Two waves.** Consider a superposition of two waves of equal amplitudes A

and with nearly the same wave numbers  $k_1$  and  $k_2$ 

$$\psi = A\cos(k_1x - \omega_1 t) + A\cos(k_2x - \omega_2 t), \qquad k_1 \approx k_2, \quad \omega_1 = \Omega(k_1), \ \omega_2 = \Omega(k_2).$$

The addition formula for cosines shows that

$$\psi = 2A\cos\left[\frac{1}{2}(k_1 - k_2)x - \frac{1}{2}(\omega_1 - \omega_2)t\right]\cos\left[\frac{1}{2}(k_1 + k_2)x - \frac{1}{2}(\omega_1 + \omega_2)t\right].$$



the combination is series of wave packets. The individual wave has wavelength

$$\lambda = \frac{2\pi}{k} \quad \left(k = \frac{k_1 + k_2}{2}\right) \quad \text{and the speed} \quad c = \frac{\omega_1 + \omega_2}{k_1 + k_2} \approx \frac{\Omega_k}{k}.$$

The modulating wave has wavelength

$$\Lambda = \frac{2\pi}{k} \quad \left(k = \frac{k_1 + k_2}{2}\right) \quad \text{and the speed} \quad C = \frac{\omega_1 - \omega_2}{k_1 - k_2} \approx \frac{d\Omega_k}{dk}$$

How can it be that the individual waves and wave packets travel with different speeds?

The individual waves move relative to the packet (in which they are in). If the phase velocity c is bigger than the group velocity C, the individual waves seem to appear from nowhere at the rear of the packet, and they seem to disappear at the front of the packet.

Each wave packet is isolated from the next one by a calm region, with no oscillations, and no energy. So, the energy has to travel with wave packets, with the group velocity C.

Although this model with two waves is quite transparent, it has some disturbing property. We usually see only one wave packet, not a series of them. To have only one wave packet, we need a linear combination of infinitely many waves.

A group of many waves with close wave numbers. Consider a superposition of infinitely many waves

$$\psi(x,t) = \int_{-\infty}^{\infty} A(k) e^{i(kx - \Omega_k t)} dk.$$

The individual harmonic waves have wave numbers near some  $k_0$  if the function A(k) has a sharp peak near  $k_0$  and vanishes away from  $k_0$ 



Since only waves with  $k \approx k_0$  are present, we can expand the dispersion law by Taylor's formula

$$\Omega(k) = \Omega(k_0) + C(k - k_0), \qquad C = \Omega'(k_0).$$

Then

$$\psi(x,t) = e^{i(k_0 x - \Omega_0 t)} \int_{-\infty}^{\infty} A(k) e^{i(k-k_0)(x-Ct)} dk.$$

The basic harmonic wave with wave number  $k_0$  has a slowly varying amplitude. The integral represents the envelop of the basic wave; it is a function of x - Ct. Hence the packet as a whole moves with the group velocity C.

*Example.* Wave-maker slowly turned on to oscillate with a single frequency  $\omega_0$  and then slowly turned off. In this way the wave-maker will create a wave packet propagating with the velocity  $C = \Omega'(k_0)$ , where the wave number  $k_0$  is determined by the frequency,  $\omega_0 = \Omega(k_0)$ . The energy from the wavemaker will propagate with the wave packet.

Stationary Phase Argument. We through a stone into a pond and thereby generate a great variety of surface waves (with various wave length). Yet after some time if we look into certain place on the pond we see quite regular waves with some specific wave length  $\lambda$ .

Suppose we observe waves at (x, t), i.e. at some place  $(x - \delta x, x + \delta x)$  at some time  $(t - \delta t, t + \delta t)$ . The coordinate x is counted from the point where

the stone was thrown, and time t is counted from the instant *when* the stone was thrown. We assume a good separation of scales:

$$x \gg \delta x \gg \lambda = \frac{2\pi}{k}, \qquad t \gg \delta t \gg \tau = \frac{2\pi}{\Omega_k}.$$

What waves will we see at (x, t)?

A naive answer to this question goes like this. The waves propagate with different velocities and thereby disperse; at (x, t) we should see almost sinusoidal wave with phase velocity c = x/t. Since c is determined by the wave number k, we can find their wave length:

$$c = \sqrt{\frac{g}{k}} = \frac{x}{t} \quad \Rightarrow k = \frac{gt^2}{x^2} \quad \text{and} \quad \lambda_c = \frac{2\pi x^2}{gt^2}$$

This reasoning turns out to be wrong: In fact, at (x, t) we will see waves 4 times longer.

For simplicity consider a 1D water surface (2D fluid). The general shape of the water surface represents a superposition of waves

$$\eta(\mathbf{x},t) = \int A_k e^{i(kx - \Omega_k t)} dk + \int B_k e^{i(kx + \Omega_k t)} dk. \quad (-\infty < k < \infty)$$

To be specific, we concentrate on the first integral; the second integral is considered similar. If x and t are large, then a small variation of k would lead to a large variation in the phase  $\theta = kx - \Omega_k t$ . So, while we integrate over k, the terms with different k will cancel each other... ... except for the point  $k_0$ of stationary phase:

$$\left. \frac{d\theta}{dk} \right|_{k_0} = 0 \quad \Leftrightarrow \quad \frac{x}{t} = \left. \frac{d\Omega}{dk} \right|_{k_0}$$

Thus, at (x,t), we will see waves with such wave number k that the group velocity

$$C(k) = \frac{d\Omega}{dk} = \frac{1}{2}\sqrt{gk} = \frac{1}{2}c$$

equals x/t. Their wavelength  $\lambda_C = \frac{8\pi x^2}{gt^2}$  is 4 times larger than the wavelength  $\lambda_c$  based on the phase speed.

Let us continue with the example of throwing a stone into a pond. Suppose after throwing a stone, we fix our gaze at a particular crest and follow it. Then we would see that the distance to the neighboring crest is continually changing. Moreover, if we continue to follow the crest, we suddenly lose sight of it. James Lighthill (in *Waves in Fluids*) writes: "It seems an optical illusion at first; but then the next crest, too, disappears! Meanwhile, crests are coming along thick and fast behind. Indeed, at the inside edge of the concentric pattern, new crests are appearing from nowhere... The suddenness with which sizable crests near the outside of the pattern disappear rules out gradual attenuation as the mechanism." The true explanation is in the fact that for the waves on deep water, the phase velocity is twice bigger than the group velocity. We will see later that the energy travels with the group velocity.

The original disturbance produces waves in some range of wave numbers  $k_1 < k < k_2$ . The typical wavelength is related to the size of the stone:  $\lambda_1 =$ 

 $2\pi/k_1$  is several times larger than the size of the stone, while  $\lambda_2 = 2\pi/k_2$  is several times smaller than the size of the stone. So, the energy is concentrated in the spatial interval

$$C_2 t < x < C_1 t$$
  $[C_1 = C(k_1), C_2 = C(k_2)].$ 

The individual waves — traveling fast, with the phase velocity c(k) = 2C(k)— disappear at the outside boundary  $C_1t$  and appear "from nowhere" at the inside boundary  $C_2t$ .

The Principle of Stationary Phase.

General math problem. Find the behavior of the integral

$$I(t) = \int_{a}^{b} f(s)e^{it\psi(s)}ds$$

for large values of the parameter  $t, t \to +\infty$ .

Since t is large, then small variations of the integration variable s would cause many oscillations of the quantity  $e^{it\psi(s)}$ , while the value of the function f(s) remains approximately the same. Therefore, contributions to the integral from different s mostly cancel each other... ... except for s from the neighborhood of special points  $s_0$  where the phase is stationary

$$\psi'(s_0) = 0$$

Suppose, for simplicity, that there is only one such point within the range of integration,  $a < s_0 < b$ . We put  $s = s_0 + \xi$ , and for small  $\xi$  we have

$$\psi(s) = \psi(s_0) + \frac{1}{2}\alpha\xi^2$$
, where  $\alpha = \psi''(s_0)$ ,

and then

$$I(t) = f(s_0)e^{it\psi(s_0)} \int_{-\infty}^{\infty} e^{it\frac{1}{2}\alpha\xi^2} d\xi.$$

In this integral, we should have integrated only over small neighborhood of the point  $s_0$ . However, — again by the stationary phase argument — we can extend the integration to the entire line  $-\infty < \xi < \infty$  (the region with large  $\xi^2$  contributes little to the integral). It is known that

$$\int_{-\infty}^{\infty} e^{i\alpha\xi^2} d\xi = \sqrt{\frac{\pi}{|\alpha|}} \frac{1 + i\operatorname{sign}(\alpha)}{\sqrt{2}} = \sqrt{\frac{\pi}{|\alpha|}} e^{i\frac{\pi}{4}\operatorname{sign}(\alpha)}$$

So,

$$I(t) = f(s_0)e^{it\psi(s_0) + i\frac{\pi}{4}\operatorname{sign}(\alpha)}\sqrt{\frac{2\pi}{t|\alpha|}}$$

**Q:** What would be the answer if the point  $s_0$  coincided with one of the limits of integration interval (a, b)?

A: Instead of the integral from  $-\infty$  to  $\infty$ , we would have integral from 0 to  $\infty$  or from  $-\infty$  to 0, and the final result had to be halved.

Apply this to the wave situation. Suppose, we fix our gaze on some place moving with constant velocity  $\nu$ . The surface shape at time t at place  $x = \nu t$ is determined by the Fourier integral

$$\eta(\nu t, t) = \int A_k e^{it[k\nu - \Omega(k)]} dk + (c.c. \text{ i.e. complex conjugated expression}).$$

So,

$$\psi(k) = k\nu - \Omega(k), \quad \Rightarrow \quad \alpha = \psi''(k_0) = -\Omega''(k_0)$$

and — according to the general result — the surface shape  $\eta(x,t)$  at large (x,t) represents a wave with wave number  $k_0$  such that

$$\Omega'(k_0) = \frac{x}{t}$$

Denote  $\Omega_0 = \Omega(k_0), \Omega_0'' = \Omega''(k_0)$ . Then

$$\eta(x,t) = A(k_0)e^{i(k_0x - \Omega_0 t)} \sqrt{\frac{2\pi}{t|\Omega_0''|}} e^{-i\frac{\pi}{4}\text{sign }\Omega_0''} + (c.c.).$$

We also see from this calculation that the wave amplitude decreases as  $t^{-1/2}$ .

If there were several stationary points, i.e. points  $k_j$  such that

$$\Omega'(k_j) = \frac{x}{t} \quad (j = 1, \dots J),$$

then each of them would make a similar contribution, with  $k_0$  replaced by  $k_j$ , and the surface shape would be the sum of all those contributions.

Note that the above consideration works only for dispersive waves:  $\Omega''(k_0)$  should be nonzero. For sound waves,  $\Omega(k) = ck$ ,  $\Omega'' \equiv 0$ , and the stationary phase principle is not applicable.

#### 2.6 Ship waves

**Phenomenon.** When a ship is moving at constant speed across deep water, the waves created by the ship are confined to a wedge of  $39^{\circ}$ .



For example you can see beautiful pictures of this phenomenon on the internet, just google "ship waves".

This angle of  $39^{\circ}$  is independent of the speed of the sheep.

It is independent of the size of the ship either. A duck creates the same wedge.

It is determined only by the fact that for gravity waves on the surface of deep water

(group velocity) = 
$$\frac{1}{2}$$
(phase velocity).

It is independent of the density of the fluid. A ship in the mercury lake would produce wedge of the same angle.

It depends on the gravity intensity neither: On a different planet we would still observe the wedge of  $39^{\circ}$ .

**Explanation.** In the frame fixed at the ship, the ship is standing in a stream, which has uniform and constant speed far from the ship. After sufficiently long time, all transients die out, and an observer from the ship should see a stationary pattern on the water surface. It is this steady pattern that we are going to examine. Thus, we assume the presence of some very small dissipative forces. In the absence of dissipative forces the wave pattern is obviously not unique. In particular, we can always add some free waves (including the free wave which is stationary in the frame fixed at the ship).

*Recall: Linear Oscillator.* To illustrate the effect of small dissipative forces, let us consider a linear oscillator

$$\ddot{u} + 2\gamma \dot{u} + \omega_0^2 u = f_0 e^{i\omega t}$$

with dissipation coefficient  $\gamma > 0$ , the own frequency  $\omega_0$ , and the forcing frequency  $\omega$  ( $\gamma$ ,  $\omega_0$ ,  $\omega$ ,  $f_0$  are constants). The general solution x(t) of this equation is the sum of two functions: (1) a particular solution, which represents harmonic oscillations with the forcing frequency  $\omega$  and (2) the general solution of the homogeneous equation. The later takes care of the initial conditions, but it dies out as  $t \to \infty$  due to the presence of small dissipation  $\gamma > 0$ , no matter how small.

In particular, when the force is time independent ( $\omega = 0$ ), the response u after sufficiently long time is also time-independent.

Suppose, the ship is traveling to the right with the speed U.



As the ship travels, at each point of its path, it generates a variety of water waves (with different wavelengths). Suppose time t ago the ship was at a point Q, and now it is at the origin O. Let us consider waves generated by the ship, when it was at the point Q. The explanation of the phenomenon consists in answering two questions.

Question 1. What wavelength should we see at point R at time t?

Answer 1. At point R at time t, we should see waves with wave number k whose group velocity is C(k) = r/t.

Question 2. Will these waves appear steady from the ship?

Answer 2. Yes, provided the phase speed c(k) equals  $U \cos \psi$ .


Indeed, consider crests of these waves. The crests would appear steady from the ship if  $U \cos \psi = c(k)$ .

The ship generates waves with different wave numbers k, and the expressions for group and phase velocities parametrically (k is the parameter) define the set of points  $R(r, \psi)$  at which (1) an observer from the ship would see some waves, and (2) those waves would appear stationary to him. The later condition implies that these waves are of significant amplitude (not just dying out transients, originated when the ship accelerated to the cruise speed U). We can easily exclude the parameter k: Since the group velocity is half of the phase velocity,

$$r = \frac{1}{2}Ut\cos\psi.$$

**Q:** What is the set of all such points?

A: Such points  $R(r, \psi)$  form a circle of diameter  $|QP| = \frac{1}{2}Ut$ 



So far, we considered only waves generated by the ship when it was at point Q, time t ago. The complete wave pattern consists of the waves generated by the ship when it was at all points behind point O, at all previous instants. At each instant -t, the ship generates waves that belong to the circle with center at S, |OS| = 3Ut/4, and radius |SQ| = Ut/4.



The union of all such circles is a wedge of angle  $2\alpha$ :

$$\sin \alpha = \frac{Ut/4}{3Ut/4} = \frac{1}{3} \quad \Rightarrow \quad \alpha = 19.5^{\circ} \quad \Rightarrow \quad 2\alpha = 39^{\circ}.$$

Wave drag. The waves generated by the ship take significant part of the power of its engine. The other part is associated with generating the boundary layer and the wake. The total drag is the sum of a wave making drag and a frictional drag. In designing a ship it necessary to find a good estimate for the power spend on wave making. It turns out that such estimate can be found from tests with small scale models.

The drag force  $F_w$  due to wave generation by geometrically similar ships can depend only on

- the size of the ship L
- its velocity U
- fluid density  $\rho$
- gravity g

So, the nondimensional quantity

$$\frac{F_w}{\rho U^2 L^2}$$
 can depend only on the Froud number  $Fr = \frac{U}{\sqrt{gL}}$ .

You can get a rough idea of this dependence from the graph



### 2.7 Group velocity again

Earlier we said that we consider the notion of the Group velocity in 5 ways. We have already considered it in three ways. In this Section we consider it in two more ways.

**Slowly varying wavelength** Let us again observe wave pattern after throwing a stone into a pond. At (x, t) we observe waves of some wavelength  $\lambda$ . This wavelength slowly varies in x and t. We can describe the entire wave pattern in the form

$$\eta(x,t) = \Re[A(x,t)e^{i\theta(x,t)}] = a(x,t)\cos[\theta(x,t) + \theta_0] \quad (A = a^{i\theta_0}),$$

The crests of the pattern at some instant  $t_{\star}$  are points where  $\theta(x, t_{\star}) + \theta_0 = 2\pi n$ (*n* is an integer). A(x, t) is the wave amplitude, which slowly varies with xand t;  $\theta(x, t)$  is the wave phase. If the pattern represented a single harmonic (sinusoidal) wave with fixed wave number k, then we would have  $\theta = kx - \Omega_k t$ . In the case of slowly varying wavelength, the wave is approximately sinusoidal, and  $k, \Omega_k$  depend on x and t:

$$k = \frac{\partial \theta}{\partial x}, \qquad \Omega_k = -\frac{\partial \theta}{\partial t}.$$

Now

$$\frac{\partial k}{\partial t} = \frac{\partial^2 \theta}{\partial t \partial x}, \quad \frac{\partial \Omega_k}{\partial x} = \frac{\partial^2 \theta}{\partial x \partial t}, \quad \Rightarrow \quad \frac{\partial k}{\partial t} + \frac{\partial \Omega_k}{\partial x} = 0$$

The quantity  $\Omega_k$  depends on x and t through k(x,t), and by the chain rule

$$\frac{\partial k}{\partial t} + \frac{d\Omega}{dk}\frac{\partial k}{\partial x} = 0$$

Recall: First Order PDE. Show that the equation

$$\frac{\partial k}{\partial t} + C \frac{\partial k}{\partial x} = 0$$

describes waves propagating with velocity C.

Indeed, the general solution of this equation is k = f(x - Ct), where the function f (of one variable) is determined by the initial condition.

In our case  $C = \Omega'$  is the group velocity. The solution is complicated by the fact that C is not a constant, but depends on k:

$$k(x,t) = f[x - C(k)t].$$

This shows that k(x,t) remains constant if  $x = C(k)t + x_0$  (with arbitrary constant  $x_0$ ). We can say that the waves of fixed wave number k propagate with the group velocity C(k).

The energy propagation. A packet of sea waves carries some energy that can bring some destruction (or useful power) to the shore. We will see now that the wave energy propagates with the group velocity.

We will calculate the energy E (per unit water surface) carried by the wave, and averaging it over one time-period  $\tau$  of the wave, find the value  $\overline{E}$ .

Then we will calculate the energy flux F (the amount of energy that passes through a vertical cross-section per unit length, per unit time), and averaging it over one time-period  $\tau$ , find the value  $\bar{F}$ .

It is natural to define the energy propagation velocity as  $\overline{F}/\overline{E}$ . We will see that this value coincides with the group velocity.

Let us calculate the energy of a water wave in the rectangle  $L_x \times L_z$  of the water surface. The total wave energy is the sum of the kinetic energy and the potential energy

$$E L_{x}L_{z} = \int dx \, dz \int_{-\infty}^{\eta} dy \, \rho \frac{u^{2} + v^{2}}{2} + \int dx \, dz \int_{-\infty}^{\eta} dy \, \rho \, g \, y$$
$$\approx \int dx \, dz \int_{-\infty}^{0} dy \, \rho \frac{u^{2} + v^{2}}{2} + \int dx \, dz \, \rho \, g \, \frac{\eta^{2}}{2}.$$

(Calculating the kinetic energy, we replaced the upper limit  $\eta$  by zero. This is consistent with our consideration of small amplitude waves:  $u^2+v^2$  is quadratic in wave amplitude, and our error in replacing  $\eta$  by 0 is of the third order.)

Suppose the water wave travels in the x-direction,  $\mathbf{k} = (p, 0)$ ,

$$\eta(x, y, z, t) = a\cos(px - \Omega_p t + \theta_0), \quad \phi(x, y, z, t) = \frac{ga}{\Omega_p} e^{|p|y} \sin(px - \Omega_p t + \theta_0).$$

From these expressions we can find  $u = \partial \phi / \partial x$  and  $v = \partial \phi / \partial y$ . Thus, the energy of the wave in the rectangle  $L_x \times L_z$  is

$$E L_x L_z = \frac{\rho g^2 a^2 |p|}{4\Omega_p^2} L_x L_z + \frac{1}{2} \rho g a^2 \cos^2(px - \Omega_p t + \theta_0) L_x L_z.$$

In the first term (kinetic energy) use the formula for the dispersion law  $\Omega_p$ . Average the second term (potential energy) over one wave period (remembering that  $\cos^2$  averages to 1/2)

$$\bar{E}L_x L_z = \frac{1}{2}\rho g a^2 L_x L_z.$$

The averaged kinetic and potential energy turned out to be same, as it should be in all cases of small amplitude oscillations.

How is the energy transported in the fluid? There are two possibilities:

- (1) The fluid is moving, and it carries with it some energy.
- (2) There are forces in the fluid that perform a work.

The first possibility for small amplitude waves turns out to be negligible. As the wave travels fluid stays at almost the same place: After one wave period, fluid particles return to their original place (almost). The second possibility gives the energy flux: The fluid at  $x < x_0$  exerts some force on the fluid at  $x > x_0$ . The work performed by this force during time  $\delta t$  is

$$\int_{-\infty}^{0} (pL_z dy) (u\delta t).$$

(Here we again replaced the upper limit  $\eta$  by 0.)

By Bernoulli's eq. 
$$p = -\rho\phi_t - \rho gy$$
, while  $u = \phi_x$ .

So,

$$\bar{F} L_z \delta t = \frac{1}{4} \rho g a^2 \frac{\Omega_p}{p} L_z \delta t$$

If the energy propagates with some velocity  $\nu$ , then the energy crossing the line  $x = x_0$  during time  $\delta t$  ( $\delta t \gg \tau$ ) is

$$\bar{E} (\nu \delta t) L_z = \bar{F} L_z \, \delta t \quad \Rightarrow \quad \nu = \frac{\bar{F}}{\bar{E}} = \frac{1}{2} \frac{\Omega_p}{p}.$$

The energy travels half as fast as the wave crests!

Since  $\Omega_p = \sqrt{g|p|}$ , we see that the energy travels with the group velocity  $C = \partial \Omega / \partial p$ .

The electric energy out of water waves? . Consider a wave of amplitude a = 1m and wave length  $\lambda = 100m$  (corresponding to the time-period  $\tau = 8s$ ). How much energy does it carry?

$$\bar{F} = 3.2 \times 10^4 \frac{W}{m}.$$

If this energy were collected along the Pacific coast of the USA, say over 1000km, then we would get power  $3.2 \times 10^4 MW$ . Compare this with usual powerplants. The most powerful plant produces  $13 \times 10^3 MW$ . It Itaipu hydroelectric plant, located on the border between Brazil and Paraguay. There are, however, considerable engineering difficulties in collecting power of water waves.

### 2.8 Waves on the surface of water of finite depth

If the water has *finite* (but uniform) depth h, the boundary condition at the bottom becomes

$$\frac{\partial \phi}{\partial y} = 0$$
 when  $y = -h$ .

Now the dispersion law of surface waves becomes

$$\Omega_{\mathbf{k}} = \sqrt{gk \tanh kh}.$$

The next graph shows the dependence of the phase velocity vs. the wave length



When  $h \to \infty$   $(kh \ll 1)$ , the dispersion law turns into the deep water dispersion law. In the opposite limit,  $h \to 0$   $(kh \gg 1)$ , we have gravity waves on the surface of shallow water; they are non-dispersive

$$\Omega_k = k\sqrt{gh},$$

like sound waves.

The graph also shows that the waves can be considered in the deep water approximation if the wavelength is less than about three depth:  $\lambda < 3h$ (instead of  $\lambda \ll h$ ).

Is compressibility important for the water wave propagation? The compressibility is negligible if the speed of sound in water,  $c_s = 1400m/s$ , is much larger than the maximal speed of water waves

$$c = \frac{\Omega_{\mathbf{k}}}{k} = \sqrt{\frac{g \tanh kh}{k}}.$$

The longest water waves have the maximal speed  $c_{\text{max}} = \sqrt{gh}$  (see the above graph)

$$\sqrt{gh} \ll c_s \quad \Leftrightarrow \quad h \ll \frac{c_s^2}{g} \approx 200 km.$$

However, the deepest places in the ocean have only about 10km depth. Hence, the compressibility of water is not important for surface water waves.

What waves do we see sitting on the beach? We mostly pay attention to the longest waves. They come with interval of about T = 8s. This wave period corresponds to the wavelength  $\lambda = 100m$  of deep water waves. However, the distance between the crests — that we see from the beach — is definitely smaller than 100m. Why?

As the waves come into gradually shallower water, the time period remains constant, but the wave length is reduced. At a depth h, the new wave number is determined by the dispersion law. For instance, at a depth h = 1m for the 8s wave, the wavelength is reduced from 100m to 25m.

### Why do waves come parallel to the beach

(irrespective of their direction in the open ocean)? As the waves come to the shore, their frequency stays constant, but their speed is gradually decreases (along with the wavelength). If the wave approaches the beach obliquely, then the parts of the wave crest closer to the shore have smaller depth and propagate slower, while parts farther from the shore have larger depth and propagate faster.



## 2.9 Surface tension

When surface tension is not neglected, the water pressure p(x, z, t) at the surface is not equal to the air pressure  $p_0$ . The difference  $p - p_0$  is due to the curvature of the surface. To calculate this pressure difference, we go via potential energy.

**Q:** What is the potential energy due to the surface tension?

A:

$$\Pi_s[\eta(x,z,t)] = \sigma \int [\sqrt{1 + (\eta_x)^2 + (\eta_z)^2} - 1] dx \, dz.$$

A small variation  $\delta\eta(x, z, t)$  of the water surface  $y = \eta(x, z, t)$  would lead to an increment  $\delta\Pi_s$  of the surface potential energy. According to thermodynamics,

 $\delta\Pi_s \equiv \Pi_s[\eta(x,z,t) + delta\eta(x,z,t)] - \Pi_s[\eta(x,z,t)] = (p-p_0)\delta V \equiv (p-p_0)\,\delta\eta\,dx\,dz.$ 

Thus,

$$p - p_0 = \frac{\delta \Pi_s}{\delta \eta \, dx \, dz} = -\sigma \left( \frac{\eta_x}{\sqrt{1 + (\eta_x)^2 + (\eta_z)^2}} \right)_x - \sigma \left( \frac{\eta_y}{\sqrt{1 + (\eta_x)^2 + (\eta_z)^2}} \right)_y,$$

which changes the dynamic boundary condition. To find the dispersion law, we need to linearize our equations, and so,

$$p - p_0 = -\sigma(\eta_{xx} + \eta_{yy}).$$

This formula for pressure difference  $p - p_0$  can be obtained easier: Take the Taylor expansion of the surface potential energy  $\Pi_s$  with respect to  $\eta(x, z, t)$ , disregard all nonlinear terms, and then find the pressure.

Thus, the linearized water wave equations are

$$\nabla^2 \phi = 0 \quad \text{in half space } -\infty < y < 0,$$
  
$$\phi_y = \eta_t \quad \text{when } y = 0 \quad (\text{kinematic boundary condition})$$
  
$$\phi_t + g\eta - \frac{\sigma}{\rho}(\eta_{xx} + \eta_{yy}) = 0 \quad \text{when } y = 0 \quad (\text{dynamic boundary condition})$$
  
$$\frac{\partial \phi}{\partial y} = 0 \quad \text{when } \quad y = -h \quad (\text{boundary condition at the bottom}).$$

This system has solutions

$$\eta = Ae^{i(px+qz-\omega t)}, \quad \phi = \Phi(y)e^{i(px+qz-\omega t)}.$$

with the frequency  $\omega$  determined by the dispersion law:

$$\omega^2 = \left(gk + \frac{\sigma}{\rho}k^3\right) \tanh(kh) \qquad (k^2 = p^2 + q^2).$$

**Q**: When are the effects of gravity and capillarity comparable? **A**:

$$gk \approx \frac{\sigma}{\rho}k^3 \quad \Leftrightarrow \quad k \approx k_m = \sqrt{\frac{\rho g}{\sigma}}.$$

For water at normal conditions

$$\sigma = 74g/s^2, \quad \rho = 1g/cm^3, \quad g = 980cm/s^2$$
  
 $\Rightarrow \quad k_m = 3.6cm^{-1}, \quad \lambda_m = \frac{2\pi}{k_m} = 1.7cm.$ 

If  $\lambda \gg \lambda_m$ , the gravity dominates, and the surface tension can be neglected. If  $\lambda \ll \lambda_m$ , the surface tension dominates, and the gravity can be neglected. This is independent of the depth h.

**Optimal depth to mimic non-dispersive waves** For experimental study of sound wave phenomena, it is advantageous to use water waves (that provide a better visualization). It is possible to choose such depth h, that there is some cancellation between the surface tension effect and the depth effect, and the dispersion is reduced. According to the dispersion law of gravity-capillary waves, we have for the phase speed c

$$c^{2} = (g + \frac{\sigma}{\rho}k^{2})k^{-1} \tanh kh = gh + (-\frac{1}{3}gh^{2} + \frac{\sigma}{\rho})hk^{2} + O(k^{4}), \quad k \to 0.$$

The  $k^2$  coefficient vanishes if

$$h = \sqrt{\frac{3\sigma}{\rho g}} = 0.5 cm$$
 for water.

The  $O(k^4)$  deviation from the non-dispersive case can be neglected within 3% accuracy if the wavelength  $\lambda > 2cm$ . This wave length corresponds to the phase speed 22cm/s. Compare this with the speed of sound (340m/s for air or 1400m/s for water. The water waves enable us to observe acoustic phenomena in slow motion.

A calm region after throwing a stone into a pond Suppose, we throw a stone into a pond. After a while, a calm circular region appears in the middle.



How fast does the middle calm region grow?

To be specific, consider surface waves on deep water. Their dispersion law corresponds to the limit  $h\to\infty$ 

$$\Omega_{\mathbf{k}} = \sqrt{gk + \frac{\sigma}{\rho}k^3}.$$



(Numbers are given for water at normal conditions).

In the presence of surface tension there is the minimum group speed and the minimum phase speed.

**Q:** Does the inside circular boundary grow with the group or phase speed?

A: The energy from the center propagates with the group velocity. All the wave energy moves faster than the minimum group velocity. So, the inside circular boundary grows with the minimal group speed C = 18cm/s. This corresponds to the wavelength  $\lambda = 2.54\lambda_m = 4.4cm$  (when  $d^2\Omega/dk^2 = 0$ ).

It is supposed here that the size of the stone is of the order 4cm to 5cm, so that it indeed excites waves in some range from below to above this length.

Since for the wavelength  $\lambda = 4.4cm$  the phase velocity exceeds the group velocity, the wave crests on the inside boundary "seem to appear from nowhere"; they emerge from the calm middle region.

Wave patterns on a stream Suppose, a log is lying across a stream. It will generate water waves. There will be waves in front of the log (upstream) and below the log (downstream).



It turns out that the waves in front of the log are short, but below the log are long. Why?

Since the forcing and the boundary conditions are stationary, the wave pattern should be stationary (all time-dependent transients die out). The water waves will be stationary if their phase velocity is opposite to the stream velocity and has the same magnitude

$$c(k) = U.$$

This equation has no solutions if  $U < c_m = 23 cm/s$ , and no waves are generated; the local disturbance dies out away from the log.

If  $U > c_m = 23 cm/s$ , this equation has two solutions —



one corresponds to longer waves, which are more like gravity waves. The other corresponds to shorter waves, which are more like capillary waves. One could expect then to see complicated, non-sinusoidal, waves. However, those waves appear in different regions.

The location of the waves is determined by the group velocity C(k). The energy of the gravity-like waves propagates with the velocity  $C(k_g) < U$ , and therefore swept downstream, below the log. At the same time, the energy of the capillary-like waves propagates with the velocity  $C(k_c) > U$ , and therefore travels upstream, in front of the log.

**Q:** The gravity waves propagate far from the log, but why don't capillary waves propagate far from the log?

A: The capillary waves are shorter and their attenuation is greater.

### 2.10 Water motion in a surface wave

The fluid speed Consider a surface wave, say wavelength  $\lambda = 100m$  and amplitude A = 1m. Its phase speed is about c = 12m/s. Estimate the maximum speed of the fluid particles.

In a surface wave

$$\eta = Ae^{i(kx-\omega t)}, \qquad \phi = Be^{|k|y}e^{i(kx-\omega t)}.$$

Since  $\phi_y = \eta_t$  (kinematic boundary condition), we have  $B = -i\omega A/|k|$ .

We consider the linear equations, and so, the real expressions for the surface deviation and the fluid potential are obtained by taking the real part of the above expressions (for simplicity assume real A)

$$\eta(\mathbf{x},t) = A\cos(\mathbf{k}\cdot\mathbf{x} - \Omega_{\mathbf{k}}t), \quad \phi(\mathbf{x},y,t) = A\frac{\omega}{k}e^{|k|y}\sin(\mathbf{k}\cdot\mathbf{x} - \omega t).$$

So,

$$u = \frac{\partial \phi}{partialx} = \omega A \frac{k}{|k|} e^{|k|y} \cos(\mathbf{k} \cdot \mathbf{x} - \Omega_{\mathbf{k}} t), \quad v = \frac{\partial \phi}{partialy} = \omega A e^{|k|y} \sin(\mathbf{k} \cdot \mathbf{x} - \Omega_{\mathbf{k}} t).$$

The fluid particles at the surface have the maximum velocity  $A\omega = (Ak)\frac{|omega|}{k}$ . It is smaller than the phase velocity by a factor  $Ak = 2\pi \frac{A}{\lambda}$ . In our example this factor is  $\approx 0.06$ .

**Orbits of fluid particles** We will show that in a surface wave over deep water, the fluid particles move in circles. The radius of the circle is proportional to the wave amplitude A. Actually, the orbits are circles with the accuracy of the order of the square of the amplitude; exact orbits are not closed. The radii also exponentially decrease with depth.

The trajectory x = X(t), y = Y(t) of a fluid particle is determined by the following ode system

$$\frac{dX}{dt} = u(X, Y, t), \quad \frac{dY}{dt} = v(X, Y, t),$$

Here the fluid velocity (u, v) is evaluated at the point (X(t), Y(t)) of the fluid particle at instant t. In a wave of small amplitude, fluid particles remain in some "small neighborhood", and at leading order the r.h.s. can be evaluated at some fixed point  $(x_0, y_0)$ , e.g. "central" to the orbit

$$\begin{aligned} \frac{dX}{dt} &= u(x_0, y_0, t) &= \omega A \frac{k}{|k|} e^{|k|y} \cos(\mathbf{k} \cdot \mathbf{x} - \Omega_{\mathbf{k}} t), \\ \frac{dY}{dt} &= v(x_0, y_0, t) &= \omega A e^{|k|y} \sin(\mathbf{k} \cdot \mathbf{x} - \Omega_{\mathbf{k}} t). \end{aligned}$$

Integrating in time, we find the trajectories of the fluid particles in a small amplitude wave. Each fluid particle during one period  $\tau$  completes a circle with center at  $(x_0, y_0)$  and radius  $Ae^{|k|y_0}$ .

Q: Do fluid particles move clockwise or counterclockwise?

A: Assume the wave travels to the right  $(\Omega_k/k > 0)$ . Then the fluid particles move clockwise.



If the water has *finite* (but uniform) depth h, the fluid particles move along ellipses. In the HW you need to find their major and minor semi-axes.

### 2.11 Waves at the interface of two fluids

The phenomenon of "dead water". Sometimes a ship gets into "dead water": Its engine works at full power, but the ship hardly moves at all. Why? The water around the ship looks calm.

**Interfacial waves** Consider two fluids: A lighter fluid, with density  $\rho_1$ , above a heavier fluid, with density  $\rho_2$ .

$\Delta \phi_1 = 0$ (y	>0)
$\phi_{1_{y}} = \eta_{t}$ (y	=0)
$\phi_{\text{lt}} + p_1/\rho_1 + g\eta = 0 \qquad (\gamma$	<i>y</i> =0)
$\eta(x,t)$	• p <sub>1</sub> X
	<b>p</b> <sub>2</sub>
Δφ <sub>2</sub> =0 (3	<i>z</i> < 0 )
$\phi_2 = \mathbf{n} \qquad (\mathbf{x})$	x=0
$\psi_{2y} = \eta_t \qquad (y)$	-0)
$\psi_{2t} + p_2/p_2 + g\eta = 0$ (1)	Y=0)

Look for the waves

$$\eta = A e^{i(kx - \omega t)},$$
  

$$\phi_1 = \Phi_1(y) e^{i(kx - \omega t)},$$
  

$$\phi_2 = \Phi_2(y) e^{i(kx - \omega t)}.$$

From the Laplace equations

$$\begin{split} \Phi_1'' + k^2 \Phi_1 &= 0 \quad \text{ and } \Phi_1(y) = C_1 e^{-|k|y}, \\ \Phi_2'' + k^2 \Phi_1 &= 0 \quad \text{ and } \Phi_1(y) = C_1 e^{+|k|y}. \end{split}$$

Then from the kinematic boundary conditions we can find  $C_1$  and  $C_2$  in terms of A

$$-C_1|k| = -i\omega A, \qquad C_2|k| = -i\omega A.$$

With the pressures found from the Bernoulli equations, the dynamic boundary condition gives

$$p_1 = p_2 \quad \Rightarrow \quad \rho_1(\dot{\phi}_1 + g\eta) = \rho_2(\dot{\phi}_2 + g\eta) \qquad (y = 0).$$

So,

$$\rho_1[C_1(-i\omega) + gA] = \rho_2[C_2(-i\omega) + gA].$$

Substituting  $C_1$  and  $C_2$  in terms of A, find the dispersion law for the interfacial waves

$$\Omega_k = \sqrt{g|k|\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}}.$$

The "dead water" explained. A ship of size l generates disturbances with wavelength  $\lambda$  of the order l ( $\lambda$  ranges from the length a few times shorter than l to the length a few times longer than l). These disturbances become waves (propagating far from the ship) if

$$U\cos\psi = c(k) = \frac{\Omega}{k}, \qquad k = \frac{2\pi}{\lambda} \approx \frac{2\pi}{l}.$$

If U < c the disturbances do not propagate; they die out away from the ship. The condition for wave generation is U > c.

For the surface gravity waves, the phase velocity c (corresponding to the wave numbers determined by the size of the ship) can be large, and so, they are not generated. At the same time the phase velocity of the interfacial waves (with the same wave number) can much smaller if the densities are almost equal. In the interfacial waves, the fluid velocity decreases exponentially away from the interface, and it is not noticeable at the water surface. So, the surface looks calm, but deep at the interface, there are big waves. The engine power can be almost entirely spent on the generation of these waves.

The phenomenon of "dead water" was reported by the Norwegian North polar expedition. Near the mouths of some of the Norwegian fjords there is a layer of fresh water over the salty water of the ocean.

"Monster" waves. What are the typical values of the factor

$$\epsilon = \sqrt{\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}}$$

in the Ocean?

A 10K temperature difference would give  $\epsilon \approx 0.02$ . A change from 0% to 5% solution of salt (by weight) would give  $\epsilon \approx 0.2$ . So, the interfacial waves travel rather slowly; their phase velocity is  $\epsilon \sqrt{g/k}$ .

Compared to the surface waves, the potential energy of the interfacial waves is greatly reduced: When the lower liquid is raised an equal amount of upper liquid of almost the same density is lowered. The kinetic energy is almost doubled due to the motion of two fluids; but the frequency is now much smaller, which reduces the kinetic energy. The average kinetic energy equals the average potential energy (as it is always the case for small amplitude oscillations). The total energy is

$$\frac{1}{2}(\rho_2 - \rho_1)g|A|^2,$$

which has a very small factor in it, and so if the motion has been set off with a reasonable amount of energy, a quite large amplitude can result.

In particular, such waves were found in Loch Ness (Scotland), where warmer water lies over colder.

### 2.12 Internal gravity waves

The buoyancy frequency (Väisälä-Brunt frequency). Consider a stratified fluid at rest

$$\mathbf{u} \equiv 0, \quad \rho = \rho_0(y).$$

Suppose a fluid element of volume  $\delta V$  is displaced from level y to level  $y + \xi$ . If the fluid is stable, the fluid element will be forced to return; it will start oscillating near its equilibrium (at level y). Assume the fluid is incompressible. Newton's second law

$$-\rho(y)\,\delta Vg + \rho(y+\xi)\,\delta Vg = \rho(y)\,\delta V\,\ddot{\xi} \quad \Leftrightarrow \quad \ddot{\xi} - \frac{g\rho_0'}{\rho_0}\xi = 0.$$

.

So the frequency of the oscillations N is given by the relation

$$N^2 = -\frac{g\rho_0'}{\rho_0}$$

(recall that stability requires  $\rho_0(y)$  to be decreasing function). The frequency N is called the buoyancy frequency, or the Väisälä-Brunt frequency.

For the ocean, the typical values of the period  $\tau = 2\pi/N$  varies from a few minutes to several hours.

**Waves.** For simplicity, consider 2D case. The frame: The axis y is vertical; the axis x is horizontal; the gravity acceleration is  $\mathbf{g} = (0, -g)$ .

The governing equations

$$\begin{array}{ll} \text{Momentum conservation:} & \rho \left[ \mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\nabla p + \rho \mathbf{g}. \\ \text{Mass conservation:} & \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0. \\ \text{Incompressibility:} & \frac{D\rho}{Dt} = 0 \end{array} \xrightarrow{\text{Due to the mass conservation}} & \nabla \cdot \mathbf{u} = 0 \end{array}$$

The exact solution — fluid at rest

$$\mathbf{u} \equiv 0, \quad \rho = \rho_0(y), \quad p = p_0(y); \quad -p'_0 - \rho_0 g = 0.$$

Linearization:

$$\mathbf{u}(x, y, t), \quad \rho = \rho_0(y) + \rho_1(x, y, t), \quad p = p_0(y) + p_1(x, y, t)$$

with  $\mathbf{u}, \rho_1, p_1$  being small (neglect quadratic terms).

$$\rho_0 \mathbf{u}_t = -\nabla p_1 + \rho_1 \mathbf{g}.$$
$$\rho_{1t} + \rho'_0 v = 0,$$
$$u_x + v_y = 0$$

We would find waves

$$u = Ue^{i(kx+ly-\omega t)}, \quad v = Ve^{i(kx+ly-\omega t)}, \quad \rho_1 = Re^{i(kx+ly-\omega t)}, \quad p_1 = Pe^{i(kx+ly-\omega t)}$$

(with all the quantities proportional to  $\exp[i(kx+ly-\omega t)]$ ; U, V, R, and P being constants) if the coefficients in the above system were constant. However, this is not the case:  $\rho_0$  depends on y.

Let us assume that the dependence  $\rho_0(y)$  is slow: The quantity  $\rho_0$  stays almost constant over a distance of one wave length in the y direction:

$$\rho_0' \ll l\rho_0$$

Then considering the coefficients as constants, we find the dispersion law of the internal gravity waves

$$\Omega_k = \frac{Nk}{\sqrt{k^2 + l^2}}.$$

The phase and the group velocities. The phase velocity gives the motion of constant phase lines

$$kx + ly - \omega t = \text{const} \quad \Rightarrow \quad \mathbf{c} = \left(\frac{\omega}{k}, \frac{\omega}{l}\right)$$

The group velocity comes from the stationary phase argument:

$$\mathbf{C} = \left(\frac{\partial \omega}{\partial k}, \ \frac{\partial \omega}{\partial l}\right).$$

For the internal gravity waves, **C** turns out to be orthogonal to the wave vector  $\mathbf{k} = (k, l)$ , i.e. parallel to the wave crests. So, the energy propagates along wave crests, in the direction orthogonal to the wave vector!

#### 2.13 Questions about waves.

1. When we consider sea waves we assume that the flow is irrotational and potential.

What is irrotational flow?

Why can we assume that the flow is irrotational at all times? (The dynamics is fixed. We should work with whatever comes from the initial conditions.)

What is a potential flow?

Is it true that any irrotational flow is potential?

What is equation for the velocity potential in the bulk of fluid? How can we have dynamics if the equation for the velocity potential is static?

2. The dynamics comes from the boundary conditions.

What is the dynamic boundary condition? How do we obtain it? What is the kinematic boundary condition? How do we obtain it?

3. What is the dispersion law? What is the dispersion law of surface gravity waves? What is the phase velocity?

- 4. Recall 5 ways to see the group velocity. In particular, what is the Stationary phase argument?
- 5. Throughing a stone into a pond. What wavelength would we observe at location x at instant t?
- 6. What are the ship waves? Would we see the same wedge on the surface of a mercury pond? Or if the gravity acceleration were twice bigger?
- 7. Waves on the surface of water of finite depth.
  What is their dispersion law?
  What waves are the fastest (short or long)?
  Waves do change when they come from the deep ocean to a beach. What stays constant (a) wave length λ, (b) wave period τ, (c) phase velocity c, or group velocity C? Explain.
- 8. Surface tension.

How does it change the boundary condition? What is the dispersion law of gravity-capillary waves on deep water? For which waves (long or short) the cappilarity effect is more important? For which waves (long or short) the gravity effect is more important? At which wave lengths these effects are both important?

- 9. We throw a stone (its diameter being about 4-5 cm) into a pond. After some time you would see a calm region. How fast does it grow? Is it determined by the phase velocity or by the group velocity?
- 10. Wave pattern on a stream near a log. Why do we see short waves up-stream and long waves dopwn-stream? When are the waves absent at all?
- 11. What are the orbits of fluid particles? How does energy propagate if the orbits are closed?Do waves carv the momentum?

# 3 Airplane Lift

Consider a solid body moving with the speed U in a fluid at rest. It is of course equivalent to the situation when the body is at rest, and the fluid has up-stream speed U. Then the flow is stationary (time-independent).



The total force  $\mathbf{F}$  acting on the body by the fluid can be decomposed into the component orthogonal to the up-stream velocity  $\mathbf{U}$  (which is called *lift*  $\mathbf{L}$ ) and the component parallel to  $\mathbf{U}$  (which is called *drag*  $\mathbf{D}$ ). The goal of flight engineers is to maximize  $\mathbf{L}$  and to minimize  $\mathbf{D}$ .

# 3.1 A sphere in ideal fluid

Consider a spere in the irrotational flow of incompressible fluid with constant density  $\rho$ .



We want to calculate the force on the sphere. Our strategy:

- 1. Find the velocity field.
- 2. Using Bernoulli's equation find the pressure along the boundary of the sphere. Then calculate the total force on the shere, integrating the pressure over the boundary.

**The first part: Velocity.** The domain outside the sphere is singly connected, and so, the flow is potential:  $\mathbf{u} = \nabla \phi$ . Let the upstream velocity be U, and the sphere have center at the origin and radius R. The potential  $\phi$  should satisfy the following boundary value problem.

$$\Delta \phi = 0 \quad \text{when} \quad r > R,$$
  
$$\nabla \phi \to (U, 0, 0) \quad \text{as} \quad r \to \infty,$$
  
$$\nabla \phi \cdot \mathbf{r} = 0 \quad \text{when} \quad r = R.$$

Let's guess the form of the solution, and then verify it. Since the sphere is completely symmetric, the solution  $\phi(\mathbf{r})$  can contain only one vector  $\mathbf{U}$  (besides  $\mathbf{r}$ ). Moreover,  $\phi(\mathbf{r})$  should depend on  $\mathbf{U}$  linearly because the boundary value problem is linear. So we are motivated to have

$$\phi = f(r)\frac{\mathbf{U}\cdot\mathbf{r}}{r} = Uf(r)\cos\theta.$$

Now substitute this solution, and find the function f(r).

The expression for the Laplacian in spherical coordinates gives

$$\Delta\phi \equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \lambda^2} = 0.$$

For our guess this equation is reduced to

$$\frac{d}{dr}(r^2f') - 2f = 0 \quad \Rightarrow \quad f = \frac{A}{r^2} + Br.$$

The boundary conditions determine the constants A and B. As  $r \to \infty$ , the flow becomes uniform  $\phi = UBr \cos \theta = Bx$ ; so, B = U. The other boundary condition can be satisfied if  $A = BR^3/2$ . It is the easiest to check this at the stagnation points (-R, 0, 0) and (R, 0, 0). Thus,

$$\phi = U\left(\frac{R^3}{2r^2} + r\right)\cos\theta.$$

Second part: Pressure. We can find pressure p from Bernoulli's equation

$$\frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} = \frac{1}{2}U^2 + \frac{p_0}{\rho}$$

 $(p_0$  is the atmospheric pressure far upstream). Then the total force on the sphere is

$$\mathbf{F} = -\int_{r=R} p\mathbf{n} dA$$

(**n** is the unit outside normal vector; for the sphere  $\mathbf{n} = \mathbf{r}/r$ ). On the surface of the sphere

$$\phi = \frac{3}{2}UR\cos\theta.$$

Since the normal velocity equals zero, Bernoulli's equation on the sphere surface becomes

$$\frac{1}{2}\left(\frac{1}{R}\frac{\partial\phi}{\partial\theta}\right)^2 + \frac{p}{\rho} = \frac{1}{2}U^2 + \frac{p_0}{\rho}.$$

Since  $\phi(\pi + \theta) = -\phi(\theta)$ , we have at symmetrical points 1 and 2



$$\left. \frac{\partial \phi}{\partial \theta} \right|_2 = \left. \frac{\partial \phi}{\partial \theta} \right|_1 \quad \Rightarrow \quad p_2 = p_1 \quad \Rightarrow \quad \mathbf{F} = 0.$$

**D'Alambert paradox.** Actually much more general result holds. In 1745, D'Alambert presented an argument that the force is zero for any body with fore-aft symmetry. At the same time, Euler gave argument for zero force even without an appeal to symmetry. He considered the momentum balance for the fluid between the body surface A and a large sphere surface S.



The total force on this fluid layer (from the body and from the fluid outside S) equals the change of momentum of this fluid layer, i.e. the momentum flux through the sphere S (the momentum flux through A is zero, since the body boundary is impermeable):

$$-\mathbf{F} + \int (-p)\mathbf{n}dS = \int \rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n})dS.$$

By Bernoulli's theorem,

$$p + \frac{1}{2}\rho |\mathbf{u}|^2 \equiv \text{const},$$

and so,

$$\mathbf{F} = -\int \frac{\rho}{2} |\mathbf{u}|^2 \mathbf{n} dS + \int \rho \mathbf{u} (\mathbf{u} \cdot \mathbf{n}) dS.$$

From a large distance, the body will look like a sphere of some small radius  $R_0$ , so that the velocity potential along the sphere surface S, is

$$\phi = U\left(\frac{R_0^3}{2r^2} + r\right)\cos\theta \quad \Rightarrow \quad \mathbf{u} = \operatorname{const} + O(R^3), \quad R \to \infty.$$

The constant term gives zero contribution to the integrals. The other term decreases as  $1/R^3$  when  $R \to \infty$ .

At the same time, the integration area is  $\propto R^2$ . Taking the limit  $R \to \infty$ , we find that  $\mathbf{F} = 0$ . Both the drag force (parallel to the velocity  $\mathbf{U}$ ) and the lift force (normal to the velocity  $\mathbf{U}$ ) vanish. However, it turns out that these arguments do not work in 2-D. In 2-D the fluid velocity behaves like  $\mathbf{u} = \text{const} + O(R) \ (R \to \infty)$ , while the integration length grows  $\propto R$ . So, the integrals become infinite.

### **3.2 2-D** flow. Stream function

All airplanes can fly due to the wings, and the wings are almost two-dimensional — their length is much larger than the cross-sectional size. It turns out that 2-D flows are crucial for the understanding of the lift force. The reason is topological. In 3-D the fluid around a body occupies a simply connected domain: Any closed curve in that domain can be continuously shrunk to a point (without leaving the domain). On the contrary in 2-D, the fluid around a body is not simply connected.



In this picture two closed curves can be shrunk to a point, and one (around the body) cannot. So, we will first study lift in two dimensional flows.

Also we will assume that the speed of the airplane is much smaller than the sound speed. Then the air can be considered incompressible. The incompressibility condition for 2-D flow reads

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

We have two functions u(x, y, t) and v(x, y, t) connected by one equation; so in fact the state of the system is determined by only one function. The equation would be automatically satisfied if there were such a function  $\psi(x, y, t)$  that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Such function is called the *stream function*. The existence of the stream function is justified similar to the existence of the potential for the irrotational flow:

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \quad \Rightarrow \quad u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}.$$

(See also HW.)

The vorticity and Euler's equation. In 2-D, the vorticity

$$\Omega = \nabla \times \mathbf{u} = (0, 0, -\Delta \psi)$$

is a scalar, and Euler's equation

$$\frac{\partial\Omega}{\partial t} + \nabla \times (\Omega \times \mathbf{u}) = 0 \quad \Leftrightarrow \quad \frac{\partial\Omega}{\partial t} + (\mathbf{u} \cdot \nabla\Omega) = 0.$$

In terms of the stream function the last equation takes the form

$$\frac{\partial \Delta \psi}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \Delta \psi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \Delta \psi}{\partial x} = 0.$$

(You show this in the HW).

Lines  $\psi(x, y, t) = const$  are the streamlines. Indeed, the fluid velocity (u, v) is orthogonal to the gradient  $\nabla \psi$ ; but  $\nabla \psi$  is orthogonal to the lines  $\psi = const$ . Hence (u, v) is parallel to the lines  $\psi = const$ .

Mass flux through an arbitrary curve  $\Gamma$  joining two points  $(x_1, y_1)$ and  $(x_2, y_2)$  equals

$$Q = \rho[\psi_2 - \psi_1].$$

This formula assumes the following agreement about signes: Positive Q means that as the fluid flows through  $\Gamma$  the point  $(x_1, y_1)$  is on the right and the point  $(x_2, y_2)$  is on the left of the flowing fluid.



$$Q = \rho \int_{(x_1,y_1)}^{(x_2,y_2)} (udy - vdx) \text{ Why? } = \rho \int_{(x_1,y_1)}^{(x_2,y_2)} \left(\frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial x}dx\right) = \rho[\psi_2 - \psi_1]$$

The fluid flow between the streamlines  $\psi = \psi_1$  and  $\psi = \psi_2$  is at rate  $\psi_2 - \psi_1$ . Usually people draw streamlines with equal incriments  $\Delta \psi$ , so that for neighboring streamlines  $\psi_2 - \psi_1 = \Delta \psi$ . Then the fluid mass flux is the same between any pair of neighboring streamlines. If the streamlines are closer together, the flow must be fast.

Let us see the streamfunctions of different simple flows.

Uniform flow with velocity U parallel to the x-axis.



Uniform shear flow parallel to the x-axis.



**Sourse-sink of fluid.** Imagine that there is a fluid source at the origin. Since the fluid is incompressible, it has to flow outwards from the origin.



Here  $(r, \theta)$  are the polar coordinates

$$x = r \cos \theta$$
,  $y = r \sin \theta$ ;  $r = \sqrt{x^2 + y^2}$ ,  $\theta = \arctan \frac{y}{x}$ 

Assume that the fluid flow is isotropic:  $\mathbf{u} = f(r)\mathbf{r}$ , where f(r) is some (so far undetermined) function of the radius. By the mass conservation

$$\int_0^{2\pi} \mathbf{u} \cdot \frac{\mathbf{r}}{r} (rd\theta) = \int_0^{2\pi} f(r) r^2 d\theta = Q \quad \text{(the total influx of the fluid.)}$$

Thus,

$$\mathbf{u} = \frac{Q}{2\pi r} \frac{\mathbf{r}}{r} = \frac{Q}{2\pi r} \begin{bmatrix} \cos\theta\\ \sin\theta \end{bmatrix}$$

What is the streamfunction  $\psi$  for this flow?

$$\frac{Q}{2\pi r}\cos\theta = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\cos\theta,$$
$$\frac{Q}{2\pi r}\sin\theta = -\frac{\partial\psi}{\partial x} = -\frac{\partial\psi}{\partial r}\cos\theta + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\sin\theta$$

(Explain these formulas, showing that

$$\frac{\partial r}{\partial x} = \cos \theta, \quad \frac{\partial r}{\partial y} = \sin \theta, \qquad \frac{\partial \theta}{\partial x} = -\frac{1}{r} \sin \theta, \quad \frac{\partial \theta}{\partial y} = \frac{1}{r} \cos \theta.$$

Multiply the first equation by  $\cos \theta$ , second equation — by  $\sin \theta$ , and add them. Also, multiply the first equation by  $\sin \theta$ , second equation — by  $\cos \theta$ , and substact them.

$$\frac{Q}{2\pi r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \\ 0 = \frac{\partial \psi}{\partial r} \qquad \Rightarrow \quad \psi = \frac{Q}{2\pi} \theta.$$

Flow in circles.



 $\begin{bmatrix} u \\ v \end{bmatrix} = F(r) \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}, \text{ where } F(r) \text{ is some (undetermined) function of the radius.}$ 

If F > 0, the fluid moves counterclockwise along the circles. The vorticity in 2-D is a scalar and equals

$$\Delta \psi = v_x - u_y = (F' \cos^2 \theta + \frac{1}{r} F \sin^2 \theta) + (F' \sin^2 \theta + \frac{1}{r} F \cos^2 \theta) = F' + \frac{1}{r} F,$$

which depends only on the radius r. For the 2-D stationary flow, the Euler equation says that  $\Delta \psi$  and  $\psi$  should be related (the vorticity  $\Delta \psi$  must be

constant along streamlines  $\psi(x, y) = \text{const.}$  So, the stream function can be an arbitrary function of the radius. Let us find the stream function when the vorticity is a constant A. Using the expression for the Laplacian in polar coordinates, we find

$$\Delta \psi \equiv \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} \equiv \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) = A \quad \Rightarrow \quad \psi = \frac{1}{4} A r^2 + B \log r + C.$$

The constant C is inconsequential and can be dropped (since the fluid velocity is given by the derivatives of the streamfunction). If B = 0, we have a solid body rotation around the origin with angula velocity A/2:

$$\left[\begin{array}{c} u\\ v \end{array}\right] = \frac{1}{2}Ar \left[\begin{array}{c} -\sin\theta\\ \cos\theta \end{array}\right]$$

If A = 0, i.e. the vorticity is zero

$$\psi = B \log r \quad \rightarrow \quad \begin{bmatrix} u \\ v \end{bmatrix} = \frac{B}{r} \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}.$$

It is called or a *line vortex* (in 3-D the origin becomes an axis z, so instead of points we have lines).

How can it be that the vortex has zero vorticity? Actually, there is vorticity: It is concentrated at the origin, where the flow is singular.

# 3.3 Magnus Effect

Imagine a circular cylinder moving in some fluid orthogonal to its axis and rotating with some angular velocity around its axis. Will it experience a lift force?

If the fluid were inviscid, its rotation should have no effect (fluid does not know about the rotation). However, in reality there is at least small viscosity, and rotating for a long time, the cylinder will make rotate the fluid around the cylinder. Now impose small stream (air jet). Will there be a lifting force?

We will see that it is possible to find lift considering irrotational, incompressible, 2-D flow around a circle.

The flow past a circle Consider a circle in the irrotational flow of incompressible fluid with constant density  $\rho$ .



We want to calculate the force on the sphere. Our strategy is as before:

1. Find the velocity field.

2. Using Bernoulli's equation find the pressure along the boundary of the sphere. Then calculate the total force on the shere, integrating the pressure over the boundary.

The first part: Velocity. In anology with the 3-D flow past the sphere

$$\phi = f(r) \frac{\mathbf{U} \cdot \mathbf{r}}{r} = U f(r) \cos \theta.$$

The expression for the Laplacian in polar coordinates gives

$$\Delta \phi \equiv \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial^2 \theta} = 0.$$

For our guess this equation is reduced to

$$\frac{d}{dr}(rf') - \frac{2}{r}f = 0 \quad \Rightarrow \quad f = \frac{A}{r} + Br.$$

The boundary conditions determine the constants A and B. As  $r \to \infty$ , the flow becomes uniform  $\phi = Br \cos \theta = Bx$ ; so, B = 1, and

$$\phi = U\left(r + \frac{A}{r}\right)\cos\theta.$$

The other boundary condition can be satisfied at the expense of the constant A; but this is a bit more dificult to check directly. Instead, we can note that the stream function for this 2-D flow is

$$\psi = U\left(r - \frac{A}{r}\right)\sin\theta.$$

Indeed, check that

$$\phi_x = \psi_y, \qquad \phi_y = -\psi_x.$$

The lines of constant  $\psi(x, y)$  are the streamlines. So the function  $\psi$  should be constant along the boundary of the circle, i.e. when r = R. An indeed, this is the case if  $A = R^2$ .

The second part: Pressure. We can find pressure p from Bernoulli's equation

$$\frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} = \frac{1}{2}U^2 + \frac{p_0}{\rho} = C(\text{const})$$

 $(p_0 \text{ is the atmospheric pressure far upstream}).$ 

Then the total force on the circle is

$$\mathbf{F} = -\int_{r=R} p\mathbf{n}Rd\theta = \begin{bmatrix} -\int_0^{2\pi} pR\cos\theta d\theta \\ -\int_0^{2\pi} pR\sin\theta d\theta \end{bmatrix}.$$

On the circular boundary

$$\phi = 2UR\cos\theta.$$

Since the normal velocity equals zero, Bernoulli's equation on the sphere surface becomes

$$\frac{1}{2} \left( \frac{1}{R} \frac{\partial \phi}{\partial \theta} \right)^2 + \frac{p}{\rho} = C.$$

So,

$$p = \rho C - 2\rho U^2 \sin^2 \theta,$$

which gives force  $\mathbf{F} = 0$ . Apparent paradox.

The flow past a circle with vortex around the circle. In the 2-D, the domain outside the circle is not simply connected. Therefore, the solution of the Laplace equation is not unique.

Consider a linear combination of the flow past a circle

$$\psi = U\left(r - \frac{R^2}{r}\right)\sin\theta$$

with the vortex (around the origin)

$$\psi = B \log r.$$

More precisely take

$$\psi = U\left(r - \frac{R^2}{r}\right)\sin\theta + B\log\frac{r}{R}.$$

The fluid velocity is described by the linear equations

$$u_x + v_y = 0$$
 (incompressible flow)  
 $v_x - u_y = 0$  (irrotational flow)  $\Leftrightarrow \Delta \psi = 0.$ 

So, the linear superposition of solutions is again a solution. The boundary condition at infinity is satisfied, since the vortex has zero velocity at infinity. The boundary condition at the circular boundary is also satisfied since the velocity field of the vortex is tangent to the circular boundary, or in other words, the streamfunction is constant along the circular boundary.

Using Bernoulli's equation, we find the pressure

$$p = \rho C - \frac{\rho}{2} \left( \frac{\partial \psi}{\partial r} \right)^2 \bigg|_{r=R} = \rho C - \frac{\rho}{2} \left( 2U \sin \theta + \frac{B}{R} \right)^2.$$

Integrating the pressure, we find the force on the body from the fluid

$$\mathbf{F} = -\int_{r=R} p\mathbf{n}Rd\theta = \begin{bmatrix} -\int_0^{2\pi} pR\cos\theta d\theta \\ -\int_0^{2\pi} pR\sin\theta d\theta \end{bmatrix} = \begin{bmatrix} 0 \\ \rho \ 2\pi B \ U \end{bmatrix}.$$

What is the meaning of the constant B?

**Circulation.** For any flow around a rigid body (of an arbitrary shape), we can consider line integrals over closed circuits

$$I = \oint_C \mathbf{u} d\mathbf{r} = \oint_C (u dx + v dy).$$

Since the fuid domain is not simply connected, there are different types of the circuits C. If the inside of C lies entirely in the fluid, I = 0. Indeed,

$$I = \int (v_x - u_y) dx dy = 0 \quad \text{(the flow is irrotational)}.$$

If C contains the circle, the integral I can be nonzero; but the integral I is the same for any circuit C that goes once around the circle.



This common value of all such integrals is called the *circulation* and denoted  $\Gamma$ . Let us calculate circulation in the flow around a circle.

Using polar coordinates

$$x = r \cos \theta$$
,  $y = r \sin \theta$ ;  $r = \sqrt{x^2 + y^2}$ ,  $\theta = \arctan \frac{y}{x}$ ,

we represent the fluid velocity in the form

$$\mathbf{u} = u\mathbf{i} + v\mathbf{j} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta$$

 $(\mathbf{e}_r \text{ and } \mathbf{e}_{\theta} \text{ are unit vectors of polar coordinates})$ . we can calculate the circulation around the circle of an arbitrary radius  $r_0$ :

$$\Gamma = \int_0^{2\pi} u_\theta r_0 d\theta = \int_0^{2\pi} \frac{\partial \psi}{\partial r} r_0 d\theta = 2\pi B.$$

So,  $B = \Gamma/(2\pi)$ , and the lift force on the circle is

$$L = \rho U \Gamma.$$

Magnus effect (see the beginning of this Section).

This result can be understood from Bernoulli's law. This will also explains the direction of force.



It turns out that this formula works for a 2-D body of any shape.

If a rigid cylindrical body is placed in a stream having the upstream velocity U, the flow is inviscid, incompressible, and irrotational, then the body experiences Lift force of aerodynamic pressure

 $L = \rho U \Gamma.$ 

# 4 Vorticity dynamics

Now we move forward a century, from Euler and Bernoulli to Kelvin and Helmholtz.

## 4.1 Persistence of circulation: Kelvin's theorem

Definition. Circulation  $\Gamma$  round a closed curve C is

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{l}.$$

Suppose the curve C is a material curve: It moves with the fluid. Then the circulation becomes a function of time t, because (1) C now depends on the instant t and (2) the velocity field  $\mathbf{u}(\mathbf{x}, t)$  changes in time.

#### Assume:

- 1. The fluid is *barotropic*: its density  $\rho$  is a function of pressure p. This is trivially the case if the fluid is incompressible.
- 2. There is no viscosity.
- 3. The force is potential  $\mathbf{F} = -\nabla \Phi$ .

**Then** the circulation

$$\Gamma(t) = \oint_{C(t)} \mathbf{u} \cdot d\mathbf{l} \tag{8}$$

remains constant.

Indeed, the curve C(t) can be parametrized by some parameter s,  $\mathbf{x} = \mathbf{X}(s,t)$ , say  $0 \le s \le 1$ . Since  $d\mathbf{l} = \frac{\partial \mathbf{X}}{\partial s} ds$ ,

$$\Gamma(t) = \int_0^1 \mathbf{u} \left( \mathbf{X}(s,t),t \right) \frac{\partial \mathbf{X}}{\partial s} ds$$

(Such an expression is often the definition of a curvilinear integral.)

Since the curve C is moving with the fluid,

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{u}.$$

Thus,

$$\frac{dC(t)}{dt} = \int_0^1 \frac{D\mathbf{u}}{Dt} \cdot \frac{\partial \mathbf{X}}{\partial s} ds + \int_0^1 \mathbf{u} \frac{\partial \mathbf{u}}{\partial s} ds.$$

In the second integral in the r.h.s. the integrand is  $\frac{1}{2}\frac{\partial}{\partial s}\mathbf{u}^2$ , and since the points corresponding s = 0 and s = 1 coincide, the integral vanishes. In the first integral on the right, the integrand — due to the Euler equation — equals

$$-\left[\frac{1}{\rho}\nabla p + \nabla\Psi\right] \cdot \frac{\partial \mathbf{X}}{\partial s},$$

and since  $\rho$  is a function of p, the integrand is also a full derivative of some function  $\phi(s)$  with  $\phi(0) = \phi(1)$ ; integration of this function also vanishes.

Thus, we prove the constancy of the circulation (8).

### Remarks.

- 1. The theorem holds even if the fluid domain is not simply connected, and the closed curve  $\Gamma$  cannot be spanned by a surface wholly lying in the fluid.
- 2. The proof requires that the viscosity is zero on the curve  $\Gamma$  (not everywhere in the fluid).

A rigid body in a stream (uniform far upstream). Suppose a rigid body is placed in stream of inviscid barotropic fluid (so that the Kelvin's theorem applies). Will the flow around the body be irrotational?

Suppose, that the flow has vorticity at some point around the body  $\mathbf{x}_0$ :  $\Omega_0 = \Omega(\mathbf{x}_0)$ . Consider a small circle with area  $\delta A$ , center at  $\mathbf{x}_0$  and oriented normal to  $\Omega_0$ . The Circulation over the boundary of this circle is

$$\Gamma = \oint \mathbf{u} d\mathbf{l} = \int (\nabla \times \mathbf{u}) \cdot \mathbf{n} dA \approx \Omega_0 \delta A$$

The fluid in the circular boundary came from upstream, where it formed some closed curve C. By Kelvin's theorem,  $\Gamma$  equals the circulation around C. But upstream the flow is uniform, and so,  $\Gamma = 0$ . Hence,  $\Omega_0 = 0$ .

How can we have circulation around the body in a stream uniform far from the body? We saw that the circulation around the body is nessary for the occurence of the lift force. See Acheson, pp. 158-159. Shedding vortisity — starting vortex.

### 4.2 Intensification of vorticity by stretching vortex tubes

**Vortex tubes.** Definition: A it vortex line is a curve which is tangent to the vorticity field  $\Omega = \nabla \times \mathbf{u}$ . In a time-dependent flow, a vortex line is defined in the form [X(s,t), Y(s,t), Z(s,t)], where t is time, and the variable s parametrises the vortex line:

$$\begin{bmatrix} \frac{\partial X}{\partial s} \\ \frac{\partial Y}{\partial s} \\ \frac{\partial Z}{\partial s} \end{bmatrix} || \Omega[X(s,t), Y(s,t), Z(s,t), t].$$

The vortex lines passing through the points of a reducible curve form a boundary of a so-called *vortex tube*.

We can cross a certain vortex tube by different surfaces. The flux of vorticity through any cross-section is the same. Indeed, apply the Gauss theorem for the volume V bounded by the surface of the vortex tube, as well as by cross-sections  $S_1$  and  $S_2$ 

$$\int_{S_1} \Omega \mathbf{n} dS - \int_{S_2} \Omega \mathbf{n} dS = \int \nabla \cdot \Omega dV = 0.$$

Here we noted that the vorticity flux through the tube boundary is zero, and the vorticity field has zero divergence. This common value of the vorticity flux is called *the strength* of the tube; it is a natural measure of its intensity. So far, we did not use the dynamics; similar tubes can be defined for any vector field with zero divergence. **Dynamics of vortex tubes** — the Helmholtz thorems. Suppose the assumptions of Kelvin's theorem hold (fluid is barotropic, inviscid, and driven by potential forces). Then

- 1. a vortex tube moves with the fluid;
- 2. the strength of a vortex tube remains constant.

These statements were discovered by Helmholtz in 1858 (120 years later after Daniel Bernoulli considered stream-tubes). Both statements follow from Kelvin's theorem, but the later appeared only in 1867 (so, Helmholtz obtained these statements in a different way).

Proof of the first statement.

Consider a material tube that initially, at instant  $t_0$ , coincides with a vortex tube. Suppose, at a later instant t, some vorticity lines go through the surface of the evolved material tube.



Then we consider a small closed circuit C(t) going around some of those vorticity lines and lying on the surface of the tube. The circulation over this circuit is non-zero (since there is vorticity flux through the circuit). At instant  $t_0$  this circut was lying on the surface of the vortex tube, and so, the circulation over  $C(t_0)$  were zero. This is in contradiction with Kelvin's thorem.

Proof of the second statement.

Consider a circut, such as  $C_1$ , composed of fluid particles that belong to the surface of the tube and encircle it once.

vortex tube



By Stokes's theorem, the strength of the vortex tube equals the circulation over  $C_1$ . By Kelvin's theorem, it remains constant in time.

**Tornado.** Suppose, the fluid is incompressible. Consider a *thin* vortex tube in which  $\Omega$  is virtually constant across any particular cross-section. First, for simplicity, assume that the tube is like a cylinder with the same cross-sectional area S, and let l be the length of a portion of the tube.



The strength of the tube is  $\Gamma = \Omega S$ . The volume of the fluid in the tube is V = Sl, which stays constant, since the fluid is incompressible. So, if the tube is stretched (*l* is increased), then the vorticity — which means rotary motion — is also increased. Actually, the vorticity is proportional to the length *l*:  $\Omega = (\Gamma/V)l$ .

In the atmospheric tornados, a vortex tube goes from the earth surface to a thundercloud. Strong thermal updraughts produce stretching of vortex tube, and the vorticity is increasing despite of the dissipation. The air pressure in the cloud is low, so the pressure in the core of the tornado vortex is also low, and hence the tornado pulls things into itself.

Demonstration: Tornado in a bottle. Gravity stretches the vortex tube. Thereby rotary motion becomes fast, and it is maintained despite of dissipation.

Spin-down of a stirred cup of tea. See Acheson, pp.164-165.

# 4.3 Vortex rings

Acheson, pp.168-172 Demonstration: Airzooka.

# 5 The viscosity

### 5.1 No-slip boundary condition.

When we considered the dynamics of inviscid fluids, we had the boundary condions

$$[\mathbf{u}_{\text{fluid}} - \mathbf{u}_{\text{solid boundary}}] \cdot \mathbf{n} = 0.$$

The fluid could slip: It could have a relative velocity tangential to the solid boundary.

Richard Feynman: The blade of fan collects a thin layer of dust. No matter how fast the fan is spinning, the dust is still there.

It turns out that

$$[\mathbf{u}_{\text{fluid}} - \mathbf{u}_{\text{solid boundary}}] = 0.$$

There is no slip. This fact is not a self-evident, but experimental evidence supports it.

#### Flow between plane surfaces.



We keep the lower plate stationary and move the upper one with some speed  $u_0$ . What force F do we need to apply to the upper plate? Obviously F is proportional to the area A of the plates. It turns out that it is also proportianal to the ratio  $u_0/d$ :

$$\frac{F}{A} = \mu \frac{u_0}{d}.$$

Actually, this is true provided (1) the velocity is small enough and (2) the fluid is Newtonian (For some complex fluids, like polymers or colloids, this relation does not hold even for small velocities.

The Navier-Stokes equations. Assume, the fluid is incompressible and Newtonian; its density  $\rho$  and the viscosity  $\mu$  being constant throughout the fluid. Then its dynamics is described by the *Navier-Stokes equations* (we will derive them later)

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{F}(\mathbf{x}, t),$$
  
$$\nabla \cdot \mathbf{u} = 0.$$

These differ from the Euler equations by the presence of viscous term  $\mu \nabla^2 \mathbf{u}$ . This difference changes the order of the PDE (from 1 to 2) and allows to satisfy the no-slip boundary condition (both — the normal and the tangential — fluid velocities should coincide with the velocity of the rigid surface). The Navier-Stokes system contains 4 equations and 4 unknown functions.

This system is basic for fluid mechanics. Though it was derived in the 19-th century, it is still unknown whether this system has a solution. The initial conditions

$$\mathbf{u}(\mathbf{r},t=0)=\mathbf{u}_0(\mathbf{r})$$

are assumed infinitely smooth, and the solution is sought on the infinite time interval, t > 0. To prove the existence of solutions (or to show that some solutions blow up in finite time) is a 1 million dollar problem. "In 2000, the Clay Mathematics Institute of Cambridge, Massachusetts (CMI) has named seven Prize Problems, focusing on important classic questions that have resisted solution over the years. The Board of Directors of CMI designated a \$7 million prize fund for the solution to these problems, with \$1 million allocated to each."

This is what is written on the web site of the institute

http://www.claymath.org/millennium/

about the importance of the fluid dynamic problem.

"Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations."

Before we derive the Navier-Stokes equations, let us consider a simple example.

Channel Flow. Consider the fluid flow between two parallel planes



Suppose that the bottom plate is at rest, while the upper plate is moving with some velocity  $u_0$  in the X direction, parallel to the plane. d is the distance between the planes. Assume that the fluid velocity is parallel to the walls, and it depends only on the coordinbate across the walls:

$$\mathbf{u} = \begin{bmatrix} u(y) \\ v = 0 \\ w = 0 \end{bmatrix}, \qquad \mathbf{F} = 0.$$
In most of situations, the Navier-Stokes equation cannot be solved analytically because this equation is nonlinear. But in this case the nonlinearity vanishes: One can calculate that

$$\frac{D\mathbf{u}}{Dt} = \begin{bmatrix} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{bmatrix} = 0.$$

So, the Navier-Stokes equation gives

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}, 0 = -\frac{\partial p}{\partial y}, \quad 0 = -\frac{\partial p}{\partial z}$$

From the second and the third equations, the pressure p depends only on x; then from the first equation,

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} = G = \text{const},$$

and therefore,

$$p = Gx + p_0,$$
  $u = \frac{G}{2\mu}y^2 + C_1y + C_2.$ 

The no-slip boundary conditions (the fluid velocity at each of the walls equals the velocity of the wall) determine the unknown constants  $C_1$  and  $C_2$ . Let us consider two important cases.

2-D Poiseuille flow. The upper plane is also at rest,  $u_0 = 0$ , and fluid flow is drived by a non-zero pressure gradient G. Then

$$u = \frac{G}{2}y(d-y)$$
 (parabolic profile).

One can calculate the flux Q of the fluid volume through a cross-section of the pipe. In the HW you need to find this flux for the 3-D pipe with circular cross-section.

This solution (with parabolic profile) exists for all possible pressure gradients G. However, when G is sufficiently large, this laminar flow becomes unstable. Absolutely different, turbulent, flow is realized in practice for large G. This was observed by Osborne Reynolds in his famous experiments in 1883. Nobody was able to calculate theoretically the dependence Q vs. G. This example is referred by Richard Feynman as an example of the Great Turbulence Problem.

**The experiment of Osborne Reynolds** In 1883 Osborne Reynolds made an experiment to study the water flow in a long tube.



Changing the height of the out flowing end would change the water velocity in the tube. Let U be the water velocity averaged over cross-section of the tube, so that the water flux (the mass of water passing per unit time through a cross-section of the tube) equals  $Q = \rho U A$ .

Reynolds also arranged that a thin stream of dye is injected in the tube. When the velocity was "small", the streak of dye was straight. This indicated that the water velocity in the tube was parallel to the tube. Fluid moved in parallel layers, forming a **laminar** flow.

When the velocity was "large", the dye spread across the tube coloring all water. The water velocity was not parallel to the tube; moreover, it was not even steady. The water flow chaotically fluctuated in space and in time. He could also see vortices. This is the **turbulence**.

The phenomenon of turbulence is of course related to the fact that fluid elements (unlike the elements of a solid medium) are not tied to their initial positions; a small force can strongly deform fluid elements.

2-D Coette flow. There is no pressure gradient, G = 0, and so,

 $u(y) = \frac{u_0}{d}y$  (the speed grows linearly from u = 0 to  $u = u_0$ ).

How to find force F that we need to apply to the upper plate in order to drag it with the speed  $u_0$ ? To answer this question, we need to introduce the notion of

#### 5.2 The stress.

As we noted in the beginning, there are two types of forces in the fluids: (1) Long-range forces and (2) Short-range forces.

1 Long-range forces. They decrease slowly with distance and can penetrate into the interior of the fluid.

Examples:

Gravity,

Electromagnetic forces (when the fluid contains charged particles),

Inertial forces (when the fluid is considered from a non-inertial frame).

Such a force is characterized by the force density  $\mathbf{g}(\mathbf{x}, t)$  per unit mass: The force acting on the fluid within a macroscopically small volume  $\delta V$  surrounding

point  $\mathbf{x}$  at instant t is

### $\mathbf{g}(\mathbf{x},t)\rho(\mathbf{x},t)\delta V$

2. Short-range forces decrease rapidly with distance. They are negligible, unless there is a direct mechanical contact between the interacting fluid elements.

How to describe such a force? It is determined not only by the point  $\mathbf{x}$  and instant t, but also by the orientation of the surface, passing through that point (i.e. by the unit normal vector  $\mathbf{n}$ ).



On the fluid element which n points away from. This force is proportoional to the surface area  $\delta A$ 

The penetration depth is supposed to be small compared with the linear dimensions of the surface element  $\delta A$ . This is possible if the fluid dynamic quantities (like  $\rho(\mathbf{x}, t)$  and  $\mathbf{u}(\mathbf{x}, t)$ ) change on much larger scale than the penetration depth.

The force per unit area,  $\Sigma(\mathbf{n}, \mathbf{x}, t)$ , is called the *stress*.

What is the stress for the inviscid fluid?  $\Sigma(\mathbf{n}, \mathbf{x}, t) = -p(\mathbf{x}, t) \mathbf{n}$ .

However, in a viscous fluid,  $\Sigma(\mathbf{n}, \mathbf{x}, t)$  is not necessarily parallel to  $\mathbf{n}$ .

The stress tensor. It turns out that the stress  $\Sigma(\mathbf{n}, \mathbf{x}, t)$  is a linear function of **n**. It is defined by  $3 \times 3$  matrix  $\sigma_{ij}$  (i, j = 1, 2, 3) (wich depends on the point  $(\mathbf{x}, t)$ , but is independent of the orientation **n**.

$$\mathbf{n} = \mathbf{e}_1 = (1, 0, 0) \quad \Rightarrow \quad \Sigma = (\sigma_{11}, \sigma_{21}, \sigma_{31}),$$
$$\mathbf{n} = \mathbf{e}_2 = (0, 1, 0) \quad \Rightarrow \quad \Sigma = (\sigma_{12}, \sigma_{22}, \sigma_{32}),$$
$$\mathbf{n} = \mathbf{e}_3 = (0, 0, 1) \quad \Rightarrow \quad \Sigma = (\sigma_{13}, \sigma_{23}, \sigma_{33}).$$

We want to show that the force on a little surface element with orientation  $\mathbf{n}$  can be expressed in the form

$$\Sigma_i(\mathbf{n}, \mathbf{x}, t) = \sigma_{ij}(\mathbf{x}, t) n_j.$$

 $\Sigma_i$  are components of the vector  $\Sigma$  (i=1,2,3);  $n_j$  are components of the vector **n** (j=1,2,3);

summation over repeated index is assumed.

This linearity implies that the infinity of the stresses (for a fixed  $(\mathbf{x}, t)$ ) can be defined by a single matrix  $\sigma_{ij}$ , which is called the *stress tensor*.

To show this linearity, we use the tetrahedron argument.

Consider a small tetrahedron in the neighborhood of a point  $\mathbf{x}$  at instant t



The large face has area  $\delta A$  and the unit normal **n** (pointing outside of the tetrahedron).

What are the forces acting on this tetrahedron element of fluid? Let  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  be the unit vectors along the axis  $X_1, X_2, X_3$  respectively. The total short-range force is

$$\Sigma(\mathbf{n})\delta A + \Sigma(-\mathbf{e}_1)\delta A_1 + \Sigma(-\mathbf{e}_2)\delta A_2 + \Sigma(-\mathbf{e}_3)\delta A_3$$

(the dependence of  $\Sigma$  on **x** and t is not displayed here, because these variables have approximately the same value for all faces).

Newton's law for the tetrahedron:

 $mass \times acceleration = long-range force + short-range force.$ 

Let the linear dimensions of the tetrahedron be l, and let it approach zero,  $l \rightarrow 0$ . The volume of the tetrahedron  $\delta V \sim l^3$ , and so are the mass and the long-range force (provided the density  $\rho$  is finite). At the same time, the area scales like  $l^2$ . Thus, the total short-range force equals zero. By Newton's third law

$$\Sigma(-\mathbf{e}_1) = -\Sigma(\mathbf{e}_1), \quad \Sigma(-\mathbf{e}_2) = -\Sigma(\mathbf{e}_2), \quad \Sigma(-\mathbf{e}_3) = -\Sigma(\mathbf{e}_3).$$

By geometric considerations

 $\delta A_1 = \mathbf{e}_1 \cdot \mathbf{n} \ \delta A = n_1 \ \delta A, \quad \delta A_2 = \mathbf{e}_2 \cdot \mathbf{n} \ \delta A = n_2 \ \delta A, \quad \delta A_3 = \mathbf{e}_3 \cdot \mathbf{n} \ \delta A = n_3 \ \delta A.$ 

(These formulas are easy to check in 2-D.) Thus,

$$\Sigma(\mathbf{n}) = \Sigma(\mathbf{e}_1)n_1 + \Sigma(\mathbf{e}_2)n_2 + \Sigma(\mathbf{e}_3)n_3$$

or in writing by components

$$\begin{bmatrix} \Sigma_1(\mathbf{n}) \\ \Sigma_2(\mathbf{n}) \\ \Sigma_3(\mathbf{n}) \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{bmatrix} n_1 + \begin{bmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{bmatrix} n_2 + \begin{bmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{bmatrix} n_3 \quad \Leftrightarrow \quad \Sigma_i(\mathbf{n}) = \sigma_{ij}n_j.$$

## 5.3 Our goal — Newton's second law for fluids.

Our goal is to derive equations which describe the motion of fluids. In particular, we wish to derive the equation which expresses Newton's second law

mass 
$$\times$$
 acceleration = Force

This equation is useful if the Force can be expressed in terms of positions and velocities of all the particles.

So, our goal now is to express the stress tensor in terms of the functions that determine the state of the fluid (in particular  $\mathbf{u}(\mathbf{x}, t)$ ,  $\rho(\mathbf{x}, t)$ , and  $p(\mathbf{x}, t)$ ).

We will assume that the stress tensor is a linear function of the velocity gradients (at the same point and at the same instant):

$$\sigma_{ij} = B_{ij} + A_{ijkl} \frac{\partial u_k}{\partial x_l}$$

(as always, repeated indices imply summation). Such fluids are called *Newto*nian.

Various symmetries lead us to conclude that

$$\sigma_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda (\nabla \cdot \mathbf{u})\delta_{ij}.$$

The coefficient  $\mu$  is called the *viscosity* (or *first viscosity*), while  $\lambda$  is called the *second viscosity*. This formula holds with excellent accuracy for many fluids — in particular, for water and air.

Let's observe that this expression indeed has nice properties.

1. When the fluid is at rest,  $\mathbf{u} \equiv 0$ , there should be no tangential stress. Why?

This means that for a fluid at rest, the stress tensor should be a diagonal matrix:

$$\sigma = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}.$$

Indeed, if, say,  $\sigma_{12} \neq 0$ , the fluid would flow,



and  $\mathbf{u} \neq 0$ .

- 2.  $\sigma_{ij}$  cannot depend on the velocity itself, since a uniform motion (with constant speed) should not be related to stress (a Galilean transformation should not change the stress).
- 3. If the fluid is in the solid body rotation, then there is no friction between fluid elements, and so, the mu- and  $\lambda$ -terms should vanish. Moreover, we can consider this fluid in the frame rotating with the fluid; then the fluid is at rest, and there should be no tangential stress (fluids cannot withstand tangential stresses).

Consider a rotation of the fluid as a whole around axis  $X_3$ . Then the velocity of the fluid

$$\mathbf{u} = \boldsymbol{\omega} \times \mathbf{x} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & 0 & \boldsymbol{\omega} \\ x_1 & x_2 & x_3 \end{vmatrix} = \begin{bmatrix} -\boldsymbol{\omega}x_2 \\ +\boldsymbol{\omega}x_1 \\ 0 \end{bmatrix}$$

Calculations give

$$\frac{\partial u_i}{\partial x_j} = \begin{bmatrix} 0 & -\omega & 0\\ \omega & 0 & 0\\ 0 & 0 & 0 \end{bmatrix},$$

and so,  $\sigma_{ij} = -p\delta_{ij}$ .

4. The relation is local in space and time:  $\sigma_{ij}$  at  $(\mathbf{x}, t)$  depends on  $\frac{\partial u_k}{\partial x_l}$  at the same  $(\mathbf{x}, t)$ .

When the fluid is incompressible,  $\nabla \cdot \mathbf{u} = 0$ , the second viscosity  $\lambda$  drops out

$$\sigma_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

**Example: Force on the upper plane in the Coette flow.** Consider again the Coette flow between two plates.

$$u(y) = \frac{u_0}{d}y \qquad \text{(the speed grows linearly from } u = 0 \text{ to } u = u_0\text{)}.$$
$$\hat{\sigma} = \begin{bmatrix} -p & u_0/d & 0\\ u_0/d & -p & 0\\ 0 & 0 & -p \end{bmatrix}.$$

If we want to calculate the force **F** on the plate from the fluid,  $\mathbf{n} = (0, -1, 0)$ , and so,

$$\mathbf{F} = A\hat{\sigma}\mathbf{n} = A \begin{bmatrix} -u_0/d \\ p \\ 0 \end{bmatrix}.$$

We need to drag the plate with the force =  $A u_0/d$ .

#### 5.4 Dynamic equation.

Now we derive the Navier-Stokes equation more rigorously: We write Newton's second law for an arbitrary material volume  $\tau$ , enclosed by the material surface S.



Recall Newton's mechanics for a system of N particles:

$$\sum_{n=1}^{N} m^{n} \mathbf{a}^{n} = \sum_{n=1}^{N} \mathbf{F}^{n} \quad (n = 1, .., N)$$

The forces  $\mathbf{F}^n$  acting on the particles in the system can be divided into the forces acting between the particles (internal forces) and the forces acting on the particles from the outside sources (outside forces). The sum of the internal forces equals zero according to Newton's third law, and so the sum of all forces in the r.h.s. can be replaced by the sum of the outside forces.

For the system of fluid particles within the material volume  $\tau$  we have

$$\sum_{n=1}^{N} m^n \mathbf{a}^n = \int \frac{Du_i}{Dt} \rho d\tau$$

equals to the total of the outside long range body forces

$$\int \rho g_i d\tau$$

and short range forces, acting along the boundary S

$$\oint \sigma_{ij} n_j dS = <$$
 Gauss theorem<sup>1</sup> >=  $\int \frac{\partial \sigma_{ij}}{\partial x_j} d\tau$ .

Thus, we have an equation with three terms, each being an integral over the volume  $\tau$ . Since  $\tau$  is arbitrary, we arrive at the equation

$$\rho \frac{Du_i}{Dt} = g_i(\mathbf{x}, t) + \frac{\partial \sigma_{ij}}{\partial x_j} \tag{9}$$

**Fluid specifics** The equation (9) is a general equation for the dynamics of any continuous medium (fluid or solid). It was obtained by Cauchy in the 1830s. For a specific continuous medium, we need to specify the expression for the stress tensor  $\sigma$ . For Netonian fluids

$$\sigma_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda (\nabla \cdot \mathbf{u})\delta_{ij}.$$

The viscosities  $\mu$  and  $\lambda$  are functions of position **x** and time *t*, because, for instance, they depend on the temperature  $T(\mathbf{x}, t)$ .

**Incompressible fluid.** We have 3 scalar equations, but 5 unknown functions  $(\mathbf{u}, p, \rho)$ . We need more equations. One of them is the mass-conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

The fifth equation would come from the energy ballance. (The Cauchy equations express the momentum ballance.)

If the fluid is incompressible, then  $\nabla \cdot \mathbf{u} = 0$ , and the second viscosity drops out. In many situations we can consider incompressible fluid as having constant density  $\rho$  throught the fluid. In addition, we can assume that the viscosity  $\mu$  is also constant throughout the fluid. In such situations, we find the Navier-Stokes equations

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu \Delta u_i + \rho g_i(\mathbf{x}, t),$$
$$\nabla \cdot \mathbf{u} = 0.$$

Thus we have a system of 4 equations for 4 unknown functions:  $u_i(\mathbf{x}, t)$ , (i = 1, 2, 3) and  $p(\mathbf{x}, t)/\rho$ . This system contains coefficients  $\rho$  and  $\mu$ , that define the properties of the fluid.

The system is essentially nonlinear, even for Newtonian fluids when the forces depend linearly on the velocity field. This is, of course, related to the fundamental fact of the fluidity. The form of the nonlinearity (in the l.h.s.) is called the *hydrodynamic nonlinearity*.

The Navier-Stokes equation should be supplemented by the *no-slip bound*ary condition: The velocity  $\mathbf{u}(\mathbf{x},t)$  of the fluid on the solid boundary should be equal to the velocity of the boundary.

According to the Navier-Stokes equations, the pressure  $p(\mathbf{x}, t)$  can be found from the velocity field. Assume for simplicity that the long-range force is zero. Take the divergence of both sides of the Navier-Stokes equation

$$\rho \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} = -\Delta p$$

#### 5.5 The Reynolds number

To be specific, let us consider a fluid flow past a cylinder (with no long range forces)

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\nabla p + \mu \Delta \mathbf{u},$$
$$\nabla \cdot \mathbf{u} = 0,$$
$$\mathbf{u} = 0 \quad \text{when} \quad x^2 + y^2 = \frac{D^2}{4},$$
$$\mathbf{u} \to (U, 0, 0) \quad \text{as} \quad \mathbf{x} \to \infty.$$

Here D is the diameter of the cylinder, and U is the stream speed at large distances from the cylinder. If we look at the equations, we see four parameters:  $\rho$ ,  $\mu$ , D, and U. It seems that to describe different regimes of the flow, we need to find four-parameter family of solutins: We need to find solutions for different values of  $\rho$ ,  $\mu$ , D, and U. However, in fact, to describe all the solutions, we need to solve this problem for different values of only one parameter.

First, note that viscosity and density appear only in the ratio  $\nu = \mu/\rho$ , which is called kinematic viscosity.

Second, suppose we measure all distances in terms of the length D, i.e. we introduce new variables x', y', z'

$$x = D x', \quad y = D y', \quad z = D z' \quad \Leftrightarrow \quad \mathbf{x} = D \mathbf{x}'.$$

In the same way, let us measure all velocities in terms of U

$$u = U u', \quad v = U v', \quad w = U w' \quad \Leftrightarrow \quad \mathbf{u} = U \mathbf{u}'.$$

Since we have fixed our units of length and velocity, our unit of time is now D/U:

$$t = t' \frac{D}{U}.$$

We also need to change the pressure

$$p = \rho U^2 p'.$$

Then we have the following problem

$$\frac{\rho U}{D/U} \frac{\partial \mathbf{u}'}{\partial t'} + \frac{\rho U^2}{D} (\mathbf{u}' \cdot \nabla') \mathbf{u}' = -\frac{\rho U^2}{D} \nabla' p' + \frac{\mu U}{D^2} \Delta' \mathbf{u}',$$
$$\nabla \cdot \mathbf{u}' = 0,$$
$$\mathbf{u}' = 0 \quad \text{when} \quad x'^2 + y'^2 = \frac{1}{4},$$
$$\mathbf{u}' \to (1, 0, 0) \quad \text{as} \quad \mathbf{x}' \to \infty.$$

All the constants condense into one parameter, called the *Reynolds number* 

$$\operatorname{Re} = \frac{\rho UD}{\mu} = \frac{UD}{\nu}.$$

Dropping the primes, we finally get

$$\begin{split} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla p + \frac{1}{\text{Re}} \Delta \mathbf{u}, \\ \nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u} &= 0 \quad \text{when} \quad x^2 + y^2 = \frac{1}{4}, \\ \mathbf{u} &\to (1, 0, 0) \quad \text{as} \quad \mathbf{x} \to \infty. \end{split}$$

It means that we can design planes using small models. Suppose we want to know characteristics of a plane of size D flying with velocity U. Instead of experimenting with the real plane, we can study the fluid flow around a small model (the model is geometrically proportional to the real plane, but has a smaller size  $D_1$ ). We can consider it flying with a different speed  $U_1$  and in a different fluid ( $\rho_1$  and  $\mu_1$ ). The real flow will be the same as the flow around the model if both Reynolds numbers are equal:

$$\frac{\rho UD}{\mu} = \frac{\rho_1 U_1 D_1}{\mu_1}.$$

Note however, that we can use this only if commpressibility can be neglected. Otherwise, a new parameter enters — the speed of sound c. Then different situations will correspond to each other only if the two flows have the same Mach numbers Ma = U/c, in addition to the same Reynolds numbers.

Remember, we used similar arguments, when we considered ship drags duer to the wave-making.

The Reynolds number shows the relative significance of the inertial and the viscous terms Consider a body with a typical size D in a stream with typical velocity U.

typical value of the inertial term 
$$= \frac{U^2}{D}$$
,  
typical value of the viscous term  $= \nu \frac{U}{D^2}$ .

The typical value of their ratio is the Reynolds number

$$\operatorname{Re} = \frac{U_0 L}{\nu}.$$

In the case of an airplane  $U \approx 100m/s$  ( $\approx 200$ miles/hour). The typical length scale  $D \approx 2m$  (the scale on which the velocity is changing significantly is about the cross-sectional size of the wing). For air (at normal conditions 1atm, 15 degrees C),  $\nu = 1.5 \times 10^{-5} m^2/s$ , and so, the Reynolds number is Re  $\approx 10^7$ . It seems that the viscous term is irrelevant and can be certainly neglected in comparison to the inertial term. That is why we considered flows of inviscid fluids. Then we got nice results, like the Kelvin's theorem, Helmholtz theorems, an so on. But we got some contradictions, in particular, the D'Alambert paradox.

It turns out that the viscous term is important in thin layers around the boundaries. If there were no viscosity, there would be no lift force for modern airplanes. In those thin layers of depth, say,  $\delta$ , the tangential component of

the fluid velocity needs to change from zero (required by the no-slip boundary condition) to the velocity of magnitude  $\approx U$  corresponding to the inviscid theory. If  $\delta$  is sufficiently small, then the inertial and the viscous term can be of the same order:

$$\frac{U^2}{D} \sim \nu \frac{U}{\delta^2} \quad \Rightarrow \quad \frac{\delta}{L} = \operatorname{Re}^{-1/2} \quad \Rightarrow \quad \delta \approx 10^{-3} m.$$

A millimeter thin layer! But it was found to be crucial. It is unclear at this point why we have chosen length scale D to estimate the inertial term, but length scale  $\delta_0$  to estimate the viscous term. If we used the scale  $\delta_0$  for both terms, we would find an incorrect result  $\frac{\delta}{L} = \text{Re}^{-1}$ .

## 5.6 The drag crisis

Let us consider the drag force on a cylinder of diameter D in a stream with velocity U far from the cylinder.

In stead of drag force F, it is better to consider the non-dimensional drag coefficient

$$C = \frac{F}{(1/2)\rho U^2 \ Dl},$$

where  $\rho$  is the fluid density, and l is the length of the cylinder. The crosssectional area perpendicular to the stream is Dl, so that  $(1/2)\rho U^2 Dl$  is the energy flux through the cross-sectional area.

The drag coefficient can be defined for an arbitrary body

$$C = \frac{F}{(1/2)\rho U^2 S},$$

where F is the drag force, and S is the cross-sectional area perpendicular to the stream.

In particular, we can consider the drag coefficient of a sphere of radius R

$$C = \frac{F}{(1/2)\rho U^2 \ \pi R^2}.$$

Consider the dependence of the drag coefficient vs. the Reynolds number (both are nondimensional).



You can see that a lot of things happen as we increase the upstream velocity (or the Reynolds number). Let's try to understand some of them.

How would we calculate the drag force F?

First, we would find the velocity field  $\mathbf{u}$  and the pressure field p from the boundary value problem

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\nabla p + \mu \Delta \mathbf{u},$$
$$\nabla \cdot \mathbf{u} = 0,$$
$$\mathbf{u} = 0 \quad \text{when} \quad x^2 + y^2 = \frac{D^2}{4},$$
$$\mathbf{u} \to (U, 0, 0) \quad \text{as} \quad \mathbf{x} \to \infty.$$

Then we would calculate the stress

$$\sigma_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Integrating the stress over the boundary, we would find the force F.

Why does the drag coefficient decrease for small Reynolds numbers? When the Reynolds number is very small, we can neglect the nonlinear (inertial) term. Considering steady flow, we drop the term with time-derivative.

$$0 = -\nabla p + \mu \Delta \mathbf{u},$$
  

$$\nabla \cdot \mathbf{u} = 0,$$
  

$$\mathbf{u} = 0 \quad \text{when} \quad x^2 + y^2 = \frac{D^2}{4},$$
  

$$\mathbf{u} \to (U, 0, 0) \quad \text{as} \quad \mathbf{x} \to \infty.$$

So, the parameter  $\rho$  drops out, and the force F should be determined by only 3 parameters  $\mu$ , D, and U. From the dimensional considerations

 $F = (\text{dimensionless constant})\mu DU.$ 

For sphere this dimensionless constant is  $3\pi$ , but to find this factor, we would need to solve the above boundary-value problem. The formula

$$F = 3\pi\mu DU$$
 (drag for a sphere)

is called the *Stokes formula*. It was used by Robert Millikan to find the charge of an electron.

The drag coefficient

$$C = \frac{(\text{dimensionless constant})\mu DU}{(1/2)\rho U^2 \pi R^2} = \frac{\text{dimensionless constant}}{\text{Re}}$$

obviously decreases with the increase of the Reynolds number (although the drag force is increasing).

When the Reynolds number increase somewhat beyond  $\text{Re} = 10^5$ , the drag coefficient drops significantly. In this range of the Reynolds numbers, the decrease in the drag coefficient is so great that the drag force itself decreases with the increasing velocity U. This is called the **drag crisis**.

What is the reason for the drag crisis?

In this range of the Reynolds numbers, a steady laminar boundary layer becomes unstable and eventually turbulent. It leads to a considerable delay (down the boundary) of the separation, so that the wake beyond the body is contracted, which implies the decrease of the drag coefficient.



Note: As  $\operatorname{Re} \to \infty$  (which is equalent to the limit of zero viscosity), the flow does not become the inviscid flow that we studied earlier. Viscosity remains to be crucial in a thin boundary layer around the solid body. Due to viscosity the velocity goes to zero in a thin boundary layer As  $\operatorname{Re} \to \infty$ , the width  $\delta$  of the boundary layer decreases.

## 6 Instabilities, Chaos, and Turbulence.

System of ODEs

$$x = F(x)$$
  $x = (x_1, x_2, \dots, x_N).$ 

Exact steady solution  $x = x_0$ :  $F(x_0) = 0$ . Linearization around  $x_0$ :  $x = x_0 + y$ 

$$\dot{x} = F(x_0) + My + (\text{nonlinear terms in } y = x - x_0), \implies \dot{y} = My.$$

Eigenvalues  $\lambda$ :

$$y = be^{\lambda t} \implies \lambda b = Mb \implies |M - \lambda M| = 0$$

## 6.1 Kelvin-Helmholtz instability

Consider interface  $y = \eta(x, t)$  between two incompressible inviscid fluids, moving relative to each other: The upper fluid has density  $\rho_1$  and velocity  $U_1$  at  $+\infty$ ; the lower fluid has density  $\rho_2$  and velocity  $U_2$  at  $-\infty$ . Assume that the flow is irrotational on both sides of the interface.



Exact steady solution

$$\eta \equiv 0, \quad \phi_1 = U_1 x, \quad \phi_2 = U_2 x.$$

Linearization around the exact solution

$$\phi_1 = U_1 x + \Phi_1, \quad \phi_2 = U_2 x + \Phi_2$$

$$\eta = A(t)e^{ikx} = Ae^{i(kx-\omega t)},$$
  

$$\Phi_1 = B_1 e^{-|k|y} e^{i(kx-\omega t)},$$
  

$$\Phi_2 = B_2 e^{|k|y} e^{i(kx-\omega t)}$$

Here k is a real wave number, and  $\omega$  is a complex frequency,  $-i\omega = \lambda$ . We have instability if  $\Im \omega > 0$ .

$$\begin{aligned} -B_1|k| &= -i\omega A + ikU_1 A,\\ B_2|k| &= -i\omega A + ikU_2 A,\\ \rho_1(-i\omega B_1 + ikU_1 B_1 + gA) &= \rho_2(-i\omega B_2 + ikU_2 B_2 + gA) \end{aligned}$$

Thus, we find the equation for the eigenvalues

$$\rho_1(\omega - kU_1)^2 + \rho_2(\omega - kU_2)^2 = (\rho_1 - \rho_2)g|k|.$$

**Rayleigh-Taylor instability.**  $U_1 = U_2 = 0$ ,  $\rho_1 > \rho_2$  (heavier fluid on top of lighter fluid).

$$\omega = \pm \sqrt{\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} g|k|}.$$

Inertial nuclear fusion. Acceleration instead of gravity. Rayleigh (1883) originally discovered this instability. Taylor (1950) recognized the significance of accelerations other than gravity.

The instability of a sheet vortex.  $\rho_1 = \rho_2$ ; discontinuity of tangential velocity.

$$\omega = k \frac{U_1 + U_2}{2} \pm ik \frac{U_1 - U_2}{2}.$$

### 6.2 Instabiliyty of parallel shear flow

See Acheson 9.5, pp. 320-325.

#### 6.3 The problem of turbulence

The development of instabilities leads to turbulence.

To be specific, let us recall the Reynolds experiment.

Why is there a problem with turbulence? In particular, is it really difficult to describe the water flow in a pipe?

The equations of Fluid Dynamics are known. So, why should we have a problem when there are powerful computers. Nevertheless, the problem is significant, which Richard Feynman called *the central* problem of modern science.

• To simulate turbulence the number of grid points should be huge. Roughly speaking

the number of grid points 
$$= \left(\frac{\text{large scale } L}{\text{small scale}}\right)^3$$
.

- We also need many time steps.
- No separation of scales. Not just microscale and macroscale; but all scales in between are present and important.
- But the hardest is *sensitivity*.

What does sensitivity mean? Suppose, we repeat the Reynolds experiment with high Re, keeping all outside conditions the same. And we measure the velocity **u** in the tube. One could expect to see the same velocity field with not a huge error, so that the experimental study would make sense. But in fact, the velocity field will be completely different: The difference of velocities will be of the same magnitude as the velocity itself. Small, "invisible", difference in the initial conditions leads to the huge difference of solutions.

Imagine that we try to simulate such system with the aid of a computer. The numerical scheme can introduce small errors, and after that — even if the further calculation is perfect (with no error) — the numerical solution will very much differ from the solution which were perfect since the very beginning.

Moreover, the errors introduced by the numerical scheme might be systematic, and the computation can be irrelevant to the reality.

At the same time it turns out that the averaged quantities are well reproducible in experiments.

For example, in the Reynolds experiment, we can look for the average velocity

$$\langle \mathbf{u} \rangle = \frac{1}{T} \int \mathbf{u}(\mathbf{x}, t) dt,$$

where T is sufficiently big, so that its further increase does not change the average.

It turns out that the averaged quantity is well reproducible in experiments: If we repeat the experiment, we find the same  $\langle \mathbf{u} \rangle$ . Moreover, we would find that the vector  $\langle \mathbf{u} \rangle$  is parallel to the tube, and its magnitude depends only on the distance r from the center.

One could also look for higher order (correlation) moments

$$\langle uv \rangle = \frac{1}{T} \int u(\mathbf{x}_1, t_1) v(\mathbf{x}_2, t_2) dt$$

. Here  $u(\mathbf{x}, t)$  and  $v(\mathbf{x}, t)$  denote some components (not necessarily the first and the second) of the velocity vector  $\mathbf{u}$  or the pressure p. Such quantity are also well reproduced in experiments.

**The ensemble average** The time average does not work if the process is not stationary, and there is no separation of time scales.

Repeating the experiment many times will produce several fields  $u^n(x,t)$ (n = 1, ..., N). The ensemble average is defined as

$$\langle u \rangle = \frac{1}{N} \sum_{n=1}^{N} u^n(x,t).$$

Similar,

$$\langle uv \rangle = \frac{1}{N} \sum_{i=n}^{N} u^n(x_1, t_1) v^n(x_2, t_2).$$

The problem of turbulence can be stated in the following way:

Describe the evolution of the averaged quantities (like  $\langle u \rangle, \langle uv \rangle, ...$ ), although we have dynamical equations for the original non-averaged quantities u, v, ... (we do not have dynamical equations for the averaged quantities).

This is the turbulence problem in a wide sense. It includes not only fluid turbulence, but also other turbulences:

- The turbulence of sea waves (rough sea).
- The turbulence of electromagnetic waves of large amplitude, so that they are nonlinear and interact with each other.
- The turbulence in plasmas.
- The turbulence of waves in solids.

#### 6.4 The closure problem

To be specific, let us consider turbulence of incompressible homogeneous fluid; its dynamics is described by the by the Navier-Stokes and mass-conservation equations

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i + \rho F_i(\mathbf{x}, t),$$
$$\frac{\partial u_i}{\partial x_i} = 0.$$

(see Section 2.2 about Conservation of momentum). However — as we have noted — often it makes no sense even to try to solve this system. We can only hope for the averaged characteristics of the flow, e.g. the average velocity field

$$\overline{\mathbf{u}}(\mathbf{x},t) = \langle \mathbf{u}(\mathbf{x},t) \rangle.$$

From the mathematical point of view, the most convenient is the ensemble average, which we assume here. In practice, we deal with time and/or space averages: The average velocity at  $(\mathbf{x}, t)$  is the average of the real fluid velocity **u** over some neighborhood of the point  $(\mathbf{x}, t)$ . The space-time average equals the ensemble average if the fluid dynamics possesses certain ergodic properties.

The Reynolds decomposition. Decompose the variable fields into the mean values and fluctuations:

$$\mathbf{u} = \overline{\mathbf{u}} + \widetilde{\mathbf{u}}, \qquad p = \overline{P} + \widetilde{p}.$$

**The Reynolds equations.** Substitute the Reynolds decomposition into the Navier-Stokes system. Using the incompressibility condition, transform  $u_j \frac{\partial u_i}{\partial x_j}$  into  $\frac{\partial}{\partial x_i} u_j u_i$ . Then take the ensemble average

$$\begin{split} \rho \frac{\partial \overline{u}_i}{\partial t} + \rho \frac{\partial}{\partial x_j} \left( \overline{u}_j \overline{u}_i + \overline{\tilde{u}_j} \overline{\tilde{u}_i} \right) &= -\frac{\partial \overline{p}}{\partial x_i} + \mu \nabla^2 \overline{u}_i + \rho \overline{F}_i(\mathbf{x}, t), \\ \frac{\partial \overline{u}_i}{\partial x_i} &= 0. \end{split}$$

Here we took advantage of the fact that the ensemble average commutes with the differentiation in time and in space. Since the averaged velocity is also incompressible, the averaged Navier-Stokes equations can be written in the form

$$\rho \frac{\partial \overline{u}_i}{\partial t} + \rho \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \mu \frac{\partial \overline{u}_i}{\partial x_j} - \rho \overline{\tilde{u}_j \tilde{u}_i} \right) + \rho \overline{F}_i(\mathbf{x}, t),$$

so that formally the averaged quantities satisfy the usual fluid dynamics equations

$$\rho \frac{D\overline{u}_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \overline{F}_i(\mathbf{x}, t), \qquad \frac{\partial \overline{u}_i}{\partial x_i} = 0$$
  
with  $\frac{D}{Dt} = \frac{\partial}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j}, \quad \sigma_{ij} = -\overline{p} + \mu \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) - \rho \overline{\tilde{u}_j \tilde{u}_i}.$ 

The only difference that now the total stress  $\sigma_{ij}$  includes an additional tensor  $\tau_{ij} = -\rho \overline{\tilde{u}_j \tilde{u}_i}$ , which is called the **Reynolds stress tensor**.

We see that the evolution of mean flow  $\bar{u}_i$  and  $\bar{p}$  depends on the second order moment  $\overline{\tilde{u}_j \tilde{u}_i}$ . This cannot be expressed in terms of the mean fields  $\bar{u}_i$  and  $\bar{p}$ , and so the averaged equations are *not closed*. We can write the evolution equation for the Reynolds stress tensor

$$\frac{\partial \overline{\tilde{u}_j \tilde{u}_i}}{\partial t} = \overline{\frac{\partial \tilde{u}_j}{\partial t} \tilde{u}_i} + \overline{\tilde{u}_i \frac{\partial \tilde{u}_j}{\partial t}} = \dots,$$

but it will contain the third order moment — the average of the product of three velocities. This is because the Navier-Stokes equation is nonlinear. The equation for the evolution of the third-order moment contains the fourth order moment and so on. The system that we would be obtained in this way is *never* closed. And this constitutes the closure problem in turbulence. There are many hypothesis how to truncate and close this system, but they are not satisfactory. In particular, engineers often use certain empirical expressions of the Reynolds stress tensor in terms of the mean velocities  $\overline{u}_i$  (and its derivatives).

Besides the mean, in many situations people are interested in the second order moment — the average of the product of velocities at two different points  $(\mathbf{x}, t)$  and  $(\mathbf{x} + \mathbf{r}, t + \tau)$ 

$$R_{ij}(\mathbf{x}, \mathbf{r}, t, \tau) = \langle u_i(\mathbf{x}, t) u_j(\mathbf{x} + \mathbf{r}, t + \tau) \rangle.$$

The Reynolds stress is expressed in terms of this function

$$\tau_{ij} = -\rho \langle \tilde{u}_j \tilde{u}_i \rangle = R_{ij}(\mathbf{x}, \mathbf{0}, t, 0).$$

The evolution of the velocity field is determined by the Navier-Stokes and the mass-conservation equations, and so, the statistical properties of the velocity field at different instants are uniquely related. Therefore, people are often satisfied with the same time correlation functions, e.g.

$$\langle u_i(\mathbf{x},t)u_j(\mathbf{x}+\mathbf{r},t)\rangle = R_{ij}(\mathbf{x},\mathbf{r},t).$$

However, the different time correlations are also important, and it is not easy to find them from the same time correlations.

### 6.5 The local structure of turbulence

It looks that we have so many problems with turbulence, and it seems a hopeless problem. In 1941 people were astonished to hear about some quantitative discovery of turbulence properties.

Consider the structure function

$$S_{ij}(\mathbf{x},\mathbf{r},t) = \langle [u_i(\mathbf{x},t) - u_i(\mathbf{x}+\mathbf{r},t)][u_j(\mathbf{x},t) - u_j(\mathbf{x}+\mathbf{r},t)] \rangle.$$

For simplicity, let us drop the sub-indices:

$$R(\mathbf{r},t) = \langle u(\mathbf{x},t)u(\mathbf{x}+\mathbf{r},t)\rangle, \qquad S(\mathbf{r},t) = \langle [u(\mathbf{x},t)-u(\mathbf{x}+\mathbf{r},t)]^2\rangle.$$

The dropping of the sub-indices can be motivated also in the following way. The problem of turbulence appears in many other contexts (not only in fluid dynamics). For instance, there is the turbulence of water waves (rough sea), and then the function u is the height  $\eta$  of the water surface  $y = \eta(x, z, t)$ .

The claim is that the structure function S represents the *local* properties of turbulence better than the correlation function. Moreover, S is *universal*: If the separation distance r is sufficiently small, the function S is independent of the particular forcing  $\mathbf{F}(\mathbf{x}, t)$  (or the boundary conditions).

Indeed, according to the Reynolds decomposition  $u = \bar{u} + \tilde{u}$ ,

$$R(\mathbf{x}, \mathbf{r}, t) = \bar{u}(\mathbf{x}, t)\bar{u}(\mathbf{x} + \mathbf{r}, t) + \langle \tilde{u}(\mathbf{x}, t)\tilde{u}(\mathbf{x} + \mathbf{r}, t) \rangle,$$
  
$$S(\mathbf{x}, \mathbf{r}, t) = [\bar{u}(\mathbf{x}, t) - \bar{u}(\mathbf{x} + \mathbf{r}, t)]^2 + \langle [\tilde{u}(\mathbf{x}, t) - \tilde{u}(\mathbf{x} + \mathbf{r}, t)]^2 \rangle$$

The mean velocity  $\bar{u}$  depends on the particular forcing or specific boundary conditions. It is supposed however, that the mean  $\bar{u}$  changes slowly in space, while  $\tilde{u}$  fluctuates fast in space. So, if the separation distance is sufficiently small, the structure function S is determined mostly by the fluctuation part, while the correlation function R strongly depends on the mean part  $\bar{u}$ , and therefore, on the particular forcing and the boundary conditions.

Let L be a typical length on which the forcing changes. The structure function is believed to be independent of the forcing details and statistically isotropic, S = S(r), if  $r \ll L$ .

What parameters can determine this function?

• Density  $\rho$ . Dimension  $\frac{kg}{m^3}$ 

- The viscosity  $\nu$ . Dimension  $\frac{m^2}{s}$
- The energy flux P (remember the example with a lake and a waterfall). Dimension of P is  $\frac{kgm^2/s^2}{kgs}$

Dimension of S is  $\frac{m^2}{s^2}$ .

**Q:** Can we find the form of this function by dimensional considerations? **A:** No: Four parameters  $r, \rho, \nu, P$ , but three dimensions m, s, kg.

$$S = P^{2/3} r^{2/3} g\left(\frac{r}{l}\right) \qquad \text{if} \quad r \ll L.$$

Here L is the integral scale, determined by the forcing. And l is the smallest scale determined by the viscosity

$$l = \left(\frac{\nu^3}{P}\right)^{1/4}.$$

It is called the **Kolmogorov microscale**. The function g of one scalar argument remains unknown.

When also  $r \gg l$ ,

$$S = CP^{2/3}r^{2/3}$$
 if  $l \ll r \ll L$ .

(Here  $C = g(\infty)$  is a nondimensional constant.) This is the major result in fluid turbulence. It is called the **Kolmogorov two-thirds law**. Many experiments in various geometries and with different fluids confirm the Kolmogorov two-thirds law.

# 7 Fluid dynamics on large scale: The dynamics of atmosphere and ocean

Our life is going in oceans and atmosphere, and it becomes increasingly important to understand their dynamics.

Examples:

- Global climate change
- Predict the path of hurricanes
- Depletion of ozone in the upper atmosphere
- Weather prediction

The science that study large scale flows on Earth (and other planets and stars) is called Geophysical Fluid Dynamics (GFD).

Two main ingredients distinguish GFD from the general Fluid Dynamics:

- Rotation
- Stratification

## 7.1 The inertial forces

To describe the the atmosphere and ocean dynamics, people use the rotating frame, fixed with Earth. This is a non-inertial frame, and besides the usual forces (like gravity and buoyancy), there are some inertial forces. Why is non-inertial frame fixed at earth better than an inertial frame?

- The velocity of the fluid relative to the earth is much smaller than the absolute velocity  $|\mathbf{u}| \ll \omega R$ , where  $\omega$  is the planetary rotation rate  $2\pi/day \approx 7 \times 10^{-5}$ , and R is the planetary radius  $\approx 6400 km$ .
- We are interested in the fluid motion relative to the earth.
- The boundary conditions are fixed at the earth.

We can describe the same phenomenon in different frames of reference. The first Newton's law states that there are inertial frames: If a particle is free of all forces, then its velocity is constant.

The second Newton's law says that in an inertial frame

 $ma = F_{total}$  where a is the acceleration of the particle with mass m.

If the total force on a particle equals zero, the velocity of the particle remains constant (i.e. its acceleration is zero). In the non-inertial frames, the total force can be nonzero even if the velocity is constant.

Example. You are in a car and consider phenomena from the frame fixed at the car. When the car accelerates, you feel some force from the seat pressing on you, though your speed (relative to the car) remains constant (zero).

We can make Newton's laws valid in non-inertial frames if we add inertial forces.

Inertial frame:  $ma = F_{total}$ .

Non-inertial frame fixed at the particle:  $F_{total} + F_{inertial} = 0$ ,

where  $F_{inertial} = -ma$  is the inertial force; it is balanced by the total real force  $F_{total}$ .

**Centrifugal force.** Suppose a particle point is **at rest in a rotating frame**. The frame rotates with constant angular velocity  $\omega$  around a fixed point, say the origin O, of an inertial frame. What is the inertial force on the particle?

In the inertial frame the particle moves with acceleration

 $\mathbf{a} = -\omega^2 \mathbf{r}$ , where  $\mathbf{r}$  is vector from the origin O to the particle.

So, in the noninertial frame the particle experiences an additional inertial force

$$\mathbf{F} = m\omega^2 \mathbf{r}.$$

Example. A water bucket on a turntable, rotating with angular speed  $\omega$ . What is the shape of the free water surface. In the rotating frame the fluid is at rest. Consider forces acting on a thin fluid element at the water surface.



Let function Z(r) define the shape of the water surface. Since fluids at rest cannot withstand any tangential stress, the water surface should be orthogonal to the sum  $\mathbf{g} + \omega^2 \mathbf{r}$ :

$$\tan \alpha = \frac{dZ}{dr} = \frac{\omega^2 r}{g} \quad \Rightarrow \quad Z(r) = \frac{\omega^2 r^2}{2g} + Z_0 \quad (Z_0 = Z(0)).$$

The fact that centrifugal force can provide an effective gravitational field is used in **centrifuges**. The centrifugal acceleration  $g' = \omega^2 r$  can be much stronger than the natural gravity acceleration g. And the rotation can be maintained for many hours. With magnetic suspensions of the rotor, it is possible to achieve centrifugal acceleration  $g' = 10^6 g$  and even larger. This can be used to separate heavier components from lighter components (even if the difference of their densities is quite small, like 1%).

## 7.2 Coriolis force

When a particle is moving in a rotating frame, it experiences an additional inertial force (compared to the particle at rest in the rotating frame). This force is independent of the position of the particle. The force is proportional to the velocity of the particle and orthogonal to the velocity-vector. This inertial force is called the *Coriolis force*  $F_{Coriolis}$ .

As an example, suppose a particle of mass m is moving radially outwards with constant speed u in the frame rotating with angular velocity  $\omega$ . The distance from the center is r = ut.

In the rotating frame, the particle is moving with constant velocity, and the total force should equal zero:  $F_{Coriolis} + F_{real} = 0$ , where  $F_{real}$  is for example the force of friction. We need to apply this force to the particle so that it could move with constant velocity.

In the inertial frame the angular momentum of the particle

$$L = mvr = m\omega r^2$$

is increasing. When the mass is close to the center, it has relatively little angular momentum, but as it moves farther out (increasing r), it has greater angular momentum. Therefore, a torque must be exerted in order to move it along the radius

Torque = 
$$F_{\text{real}}r = \frac{dL}{dt} = 2m\omega r\frac{dr}{dt}$$
.

The Coriolis force is therefore the sidewise force

$$F_{Coriolis} = -2m\omega u.$$

However, the above argument is not general enough:

J

j(t

- What if the particle moves not radially?
- What if the particle moves with some acceleration relative to the rotating frame?

**General argument.** In the inertial frame the motion of the particle is described by the vector-function



i(t)

х

► X

ωt

Differentiating in time t, we find the transformation of velocities

$$\mathbf{U} = \dot{x}\,\mathbf{i} + x\,\mathbf{i} + \dot{y}\,\mathbf{j} + y\,\mathbf{j} = \mathbf{u} + \boldsymbol{\omega} \times \mathbf{r},$$

where  $\mathbf{u} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$  is the velocity of the particle in the rotating frame. Here we have used the relations  $\mathbf{\dot{i}} = \omega \times \mathbf{i}$  and  $\mathbf{\dot{j}} = \omega \times \mathbf{j}$ .

Differentiating again, we find the transformation of accelerations

$$\mathbf{A} = \ddot{x}\,\mathbf{i} + 2\dot{x}\,\dot{\mathbf{i}} + x\,\ddot{\mathbf{i}} + \ddot{y}\,\mathbf{j} + 2\dot{y}\,\dot{\mathbf{j}} + y\,\ddot{\mathbf{j}} = \mathbf{a} + 2\,\omega\times\mathbf{u} + \omega\times[\omega\times\mathbf{r}].$$

Here the first term  $\mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$  is the acceleration of the particle in the rotating frame; the last term is the

centripetal acceleration = 
$$\omega \times [\omega \times \mathbf{r}] \equiv -\omega^2 \mathbf{r}$$
,

which gives the

centrifugal force = 
$$-m\omega \times [\omega \times \mathbf{r}] = m \omega^2 \mathbf{r};$$

and the second term  $2\omega \times \mathbf{u}$  is the *Coriolis acceleration*, which gives the

Coriolis force = 
$$-2m \ \omega \times \mathbf{u}$$
.

There is no additional inertial force due to the acceleration  $\mathbf{a}$  of the particle in the rotating frame.

When the rotation is counterclockwise,  $\omega = (0, 0, \omega > 0)$ , the Coriolis force always tries to deflect the particle to the right.

**The Foucault pendulum.** In 1851 J.B.L. Foucault demonstrated the Earth's rotation with the aid of a large pendulum. You can see the Foucault pendulum, for instance, in the Brigham Young University (along with several other cool demonstrations).

To be a Foucault pendulum, a pendulum should possess the following two properties. (1) The pendulum should be large, so that frictional effects are negligible, and the pendulum could oscillate for a long time. (2) The suspension should allow free rotation, so that the pendulum can freely change the plane of oscillation.

The Foucault pendulum oscillates in the plane that rotates relative to the earth. If the pendulum were suspended at the North or South pole, we would observe that the earth is rotating beneath the pendulum oscillating in a fixed plane.

In the frame fixed at the earth, the pendulum experiences the Coriolis force. In the Northern hemisphere, the Coriolis force tries to deflect the bob to the right (independent of the direction of swing).

Demonstration. Vortex in a sink.

#### 7.3 Motion under the action of the Coriolis force.

**Geopotential.** In the absence of rotation gravity would keep a planet in shape of a sphere. The outward pull caused by the centrifugal force distorts this spherical equilibrium, and the planet assumes a shape, slightly flattened at the poles.



The sum of the gravitational force and the centrifugal force defines the direction of the local vertical. On the earth, for example, the flattening is very slight, because the gravity by far exceeds the centrifugal force. The terrestrial equatorial radius is 6378 km, and its polar radius is 6357 km.

The centrifugal force is potential; it can be expressed as a gradient of some scalar function:

$$\omega \times [\omega \times \mathbf{r}] = -\frac{1}{2} \nabla |\omega \times \mathbf{r}|^2.$$

which together with gravity forms the gepotential

$$\Phi = (\text{gravitational potential}) - \frac{1}{2} |\omega \times \mathbf{r}|^2.$$

$$\mathbf{g} = -\nabla \Phi$$
 is the local gravity acceleration;

which defines the local vertical. If the earth were covered by water (and the water were in equilibrium), the surface of the earth would be a geopotential surface  $\Phi = \text{const.}$  To the first approximation the earth is an oblate spheroid: A cross-section through the earth is an ellipse. Its polar radius (or semiminor axis) is smaller than its equatorial radius (or semimajor axis) by 21km. Due to inhomogeneous distribution of mass in the earth, the geopotential surface departs from this ellipsoid; the largest departure is a depression of "depth" about 100 m just to the south of India. For comparison, the fluid dynamic disturbances are of a few meters.

The local coordinate frame:

axis z is up, in the direction  $-\mathbf{g}$ ; plane (x, y) is horizontal, tangent to the geopotential surface;

axis y is Northwards, the plane (y, z) contains the axis of the earth rotation; axis x is Eastwards, orthogonal to axes y and z.



In this coordinate system, the velocity vector is (u, v, w), and the vector of earth's rotation is  $(0, \omega \cos \lambda, \omega \sin \lambda)$ , where  $\lambda$  is the latitude of the local place, and  $\omega$  is the angular velocity of the earth,  $\omega = 2\pi/\text{day}$ .

A free particle on a horizontal plane. In the frame rotating with the earth

$$m\mathbf{a} = (\text{gravity force}) + (\text{centrifugal force}) - 2m \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega \cos \lambda & \omega \sin \lambda \\ u & v & 0 \end{vmatrix}.$$

Note: w = 0 because the particle moves in the horizontal plane. Consider projection of the last vector equation on the horizontal plane

 $\dot{u} = 2v\omega\sin\lambda, \qquad \dot{v} = -2u\omega\sin\lambda.$ 

The horizontal component of the earth angular velocity drops out. Only the component orthogonal to the geopotential surface enters the equations of motion. It is denoted  $f(y) = \omega \sin \lambda$  and called the *Coriolis parameter*. So,

$$\begin{cases} \dot{u} = fv, \\ \dot{v} = -fu \end{cases} \Rightarrow \quad \ddot{u} + f^2 u = 0 \quad \Rightarrow \quad \begin{aligned} u = V \sin(ft + \phi), \\ v = V \cos(ft + \phi), \end{aligned}$$

where V and  $\phi$  are two constants of integration,  $(V \sin \phi, V \cos \phi)$  is the initial velocity.

The magnitude of particle's velocity remains unchanged:  $\sqrt{u^2 + v^2} = V$ , but the velocity direction changes in time. To document this curving effect, it is most instructive to derive the trajectory [x(t), y(t)] of the particle.

$$\begin{cases} \dot{x} = u, \\ \dot{y} = v \end{cases} \Rightarrow \begin{aligned} x = x_0 - \frac{V}{f}\cos(ft + \phi), \\ y = y_0 + \frac{V}{f}\sin(ft + \phi), \end{aligned}$$

where  $x_0$  and  $y_0$  are additional constants of integration (they are determined by the initial position of the particle). We see that the trajectory of the particle is a circle

$$(x - x_0)^2 + (y - y_0)^2 = \left(\frac{V}{f}\right)^2$$

with radius V/|f|.

Where does the particle turn? Does it rotate clockwise or counterclockwise? In the Northern hemisphere, the Coriolis parameter is positive, and the particle turns to the right, rotating clockwise. In the Southern Hemisphere, the Coriolis parameter is negative, and the particle turns to the left, rotating counterclockwise.

What is the period of particle's rotation? The particle covers the full circle in time  $T = 2\pi/f$ , called *inertial period*. The circular motion of a free particle is called *inertial oscillation*. The inertial period ranges from 12 hours at the poles to infinity at the equator.

Example. On a perfectly smooth and frictionless hockey field at Dartmouth College ( $\lambda = 43^{\circ}38'$ ), what should be the velocity of a puck so that it would perform an inertial circle of diameter equal to the field width (26 m) ?

$$\begin{split} \lambda &= 43^{\circ}38' \quad \Rightarrow \quad f \approx 3.5 \times 10^{-5} 1/s, \\ R &= 13m = \frac{V}{f} \quad \Rightarrow \quad V = fR \approx 4.5 \times 10^{-4} m/s. \end{split}$$

#### 7.4 Geostrophic flow

Euler's equation for the oceans and atmosphere First of all we should note that, in many respects, the viscosity is not important for the large scale motions considered in Geophysical Fluid Dynamics. Indeed, let us estimate the Reynolds number  $Re = UD/\nu$ . Typically

$$\nu \sim 10^{-5} (\text{air}) \text{ and } 10^{-6} (\text{seawater}) m^2 s^{-1},$$
  
 $U \sim 1 (\text{air}) \text{ and } 0.1 (\text{seawater}) m/s,$   
 $D \sim 100m, \text{that give } Re \sim 10^7.$ 

So, we neglect viscosity. In addition we should note that the flow is turbulent

In the local coordinates, the fluid dynamics obeys the equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\ \omega \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g},$$

where  $\omega = (0, \omega \cos \lambda, \omega \sin \lambda)$  is the vector of angular velocity of earth's rotation.

**Q:** Where is the centrifugal force?

Let us estimate the order of magnitude of different terms in this equation. For geophysical motions, the velocity has magnitude  $u \sim 1m/s$ , and it changes on scale  $L \sim 50 km$ . So,

the inertial term 
$$\sim \frac{(1m/s)^2}{5 \times 10^4 m} = 2 * 10^{-5} \frac{m}{s^2}$$
,  
the Coriolis term  $\sim 2 \left(\frac{2\pi}{243600} s^{-1}\right) \frac{1m}{s} \approx 10^{-4} \frac{m}{s^2}$ .

Compare this with the gravity term  $g \approx 10 \frac{m}{s^2}$ .

Does it make sence to consider the inertial and Coriolis terms if they are so much smaller than the gravity?

Yes, it does. Two reasons:

1. The gravity is almost completely compensated by the pressure term

$$\nabla p = \rho \mathbf{g}$$
 — hydrostatic ballance.

 The projection of g on the horizontal plane is zero. So, we write the Euler equation in components

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - 2\omega_z v + 2\omega_y w = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + 2\omega_z u = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - 2\omega_y u = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g.$$

This should be supplemented by the mass conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

Can we assume that the fluid is incompressible?

Mach number 
$$= \frac{U}{c} \sim \frac{1m/s}{1500m/s}$$
(ocean)  $\ll 1$ .

We suppose that the fluid is incompressible. (The density of a fluid element does not change as it moves; but the density can be non-constant throughout the fluid).

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$
 and  $\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0.$ 

**Shallow water approximation.** We describe the dynamics of a thin fluid layer on the earth



Length scales L and H: The flow *changes* significantly over the length of the order L in the horizontal direction and over the length of the order H in the vertical direction.

#### Shallow water approximation: $H \ll L$ .

Geophysical flows are typically confined to domains that are much wider than they are thick. For example, the atmospheric layer that determines our whether is only 10 km thick, yet cyclones and anticyclones spread over thousands of kolometers. Ocean currents, generally confined to the upper hundred metters of the water column, may extend over tens of kilometers (or more, up to the width of the ocean basin).

According to the mass conservation equation,

$$O(\frac{U}{L}) + O(\frac{U}{L}) + O(\frac{W}{H}) = 0 \quad \Rightarrow \quad W \ll U.$$

So, Geophysical flows are almost two dimensional.

Hydrostatic ballance. We already know that

$$\rho = \rho_0(z) + \tilde{\rho}(x, y, z, t), \quad p = p_0(z) + \tilde{\rho}(x, y, z, t) \quad \text{where } p_0(z) \text{ and } \rho_0(z) \text{ are in hydrostatic ballance}$$

So the Euler equation can be written in the form

$$\rho\left(\frac{D\mathbf{u}}{Dt} + 2\,\boldsymbol{\omega} \times \mathbf{u}\right) = -\nabla \tilde{p} + \tilde{\rho}\mathbf{g}.$$

When  $\tilde{\rho} < 0$ , the term is the *buoyancy* force, directed upwards.

It turns out that the gravity dominates the z-component of the momentum equation (Euler's equation), and

$$\frac{\partial p}{\partial z} = -\rho g.$$

Even the perturbations  $\tilde{p}(x, y, z, t)$  and  $\tilde{\rho}(x, y, z, t)$  are in hydrostatic ballance  $\frac{\partial \tilde{p}}{\partial z} = -\tilde{\rho}g$ .

Indeed,

$$\tilde{p} \sim \frac{1}{2} \rho U^2 \quad \Rightarrow \quad \frac{U}{H}$$

d

 $u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \sim U\frac{W}{L} + \frac{W^2}{H} \sim \frac{UW}{L}$  (the later estimate is due to the mass conservation equation

$$\frac{\partial w}{\partial t} \sim \frac{W}{L/U} \sim \frac{UW}{L}, 2\omega_y usim 2\omega_y$$

We see that

$$\tilde{p}$$
 dominates  $\frac{D\mathbf{u}}{Dt}$  if  $UL \gg WH$   
(which is guaranteed by the shallow water approximation);  
 $\tilde{p}$  dominates  $2\omega_y u$  if  $H \ll \frac{U^2}{H} \sim \frac{(1m/s)^2}{7 \times 10^{-5} 1/s} \sim 10^4 m.$ 

The importance of the vertical component of the earth rotation. Recall the motion of a particle on a horizontal plane. The particle was rigidly placed in the plane, its vertical velocity w = 0. Fluid in a layer on the horizontal plane can move in any direction, and w can be nonzero. However,  $W \ll U$ , and the motion is almost in the plane. So, in the *x*-momentum equation, we neglect the Coriolis term with the horizontal component of the earth angular velocity.

Thus, we find

$$\begin{split} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + -f(y)v &= -\frac{1}{\rho} \frac{\partial p}{\partial x}, \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}f(y)u &= -\frac{1}{\rho} \frac{\partial p}{\partial y}, \\ \frac{\partial p}{\partial z} &= -\rho g. \end{split}$$

#### Rossby number.

$$\frac{\text{typical magnitude of the advective (nonlinear) term}}{\text{typical magnitude of the Coriolis (linear) term}} \sim \frac{U^2/L}{fU} = \frac{U}{fL} = Ro.$$

If  $Ro \gg 1$  than the rotation of the earth is not important and can be neglected. If  $Ro \ll 1$ , the earth rotation is the dominant factor, and the Coriolis force is the dominant force.

The Rossby number can be understood as the ratio of typical fluid frequency 1/T (where T = L/U is the typical period) to the double frequency of the earth rotation at a specific lattitude  $f = 2\omega \sin \lambda$ :

$$Ro = \frac{U/L}{2\omega \sin \lambda}.$$

If the phenomenon occurs during time T, much smaller than 24 hours, the rotation can be neglected. In particular, in that demonstration with draining water, the rotation is absolutely unimportant.

Obviosly, in most engineering applications, the effects of rotation can be ignored.

Examples.

Jet:  $U \sim 100m/s$ ,  $L \sim 10m$ ,  $f \sim 10^{-4} \Rightarrow Ro \sim 10^5$ Turbine in hydro-power plant:  $U \sim 10m/s$ ,  $L \sim 1m$ ,  $f \sim 10^{-4} \Rightarrow Ro \sim 10^5$ However, we see that the Rossby number can be quite small for large scale geophysical flows. For such flows the earth rotation is of principal importance. Examples. Ocean currents:  $U \sim 0.1m/s$ ,  $L \sim 10km$ ,  $f \sim 10^{-4} \Rightarrow Ro \sim 0.01$ Atmosphere:  $U \sim 10m/s$ ,  $L \sim 100km$ ,  $f \sim 10^{-4} \Rightarrow Ro \sim 0.001$ 

Large scale motions experience strong influence of the Coriolis force (even if their velocity is small).

**Geostrophic ballance.** If the Rossby number is small, we can neglect the inertial terms. We also neglect the terms with time derivative (assuming that the time scale is of the order T = L/U). Then the horizontal momentum equations give

$$-f(y)v = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$f(y)u = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

This is the *geostrophic ballance*: The Corriolis force (dominating the inertial term) ballances the pressure force.

The geostrophic velocity

$$u = -\frac{1}{\rho f} \frac{\partial p}{\partial y}, \qquad v = +\frac{1}{\rho f} \frac{\partial p}{\partial x}$$

is perpendicular to the pressure gradient

$$\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right).$$

The fluid particles are not cascading down-gradient from high to low pressures, as in nonrotating fluid, but, instead, are navigating along the lines of constant pressure, called *isobars*.

What is the direction of the fluid flow along isobars?

For an arbitrary vector (a, b)



In the Northern hemisphere, the fluid flows with the high pressure on their right.



In the Southern hemisphere, the fluid flows with the high pressure on their left.

1 flag = 50 knots 1 barb = 10 knots 1/2 barb=5knots 1 knot (nautical mile per hour)  $\approx 0.5m/s$ 

Since the fluid velocity is orthogonal to the pressure gradient, no pressure work is performed; and once initiated, the flow can persist without energy source.

The function f(y) is a "slow" function of the lattitude:

$$f = 2\omega \sin \lambda;$$
  $\frac{1}{f}\frac{df}{dy} = \tan \lambda \frac{1}{R};$   $\frac{1}{f}\Delta f = \frac{1}{f}\frac{df}{dy}L = \tan \lambda \frac{L}{R}.$ 

The *f*-plane approximation considers the Coriolis parameter as a constant:  $f(y) = f_0$ . The  $\beta$ -plane approximation takes 2 first terms of Taylor's series for the Coriolis parameter  $f(y) = f_0 + \beta y$ .

In the f-plane approximation the horizontal velocity in the geostrophic ballance satisfies the mass conservation equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

In the  $\beta$ -plane approximation, we need some small vertical velocity in order to satisfy the mass conservation.