Lecture Notes
VIGRE Minicourse on Nonconvex Variational Problems and Applications
University of Utah, Salt Lake City, May 23, 2005

Some Variational Problems in Geometry I

Andrejs Treibergs
University of Utah

Abstract. In this lecture we describe some elementary varitional problems from geometry and mention some higher dimensional generalizations. We begin by discussing two problems for embedded plane curves, the reverse isoperimetric inequality for curves of bounded curvature and the deformation of a planar elastic ring under hydrostatic pressure. These problems illustrate how topological and geometric conditions of the problem, as well as the coordinate invariance of the quantities involved tend to make the problems inherently nonlinear and often nonconvex. Arguments combine analytic and geometric considerations.

Many of the main problems of differential geometry are variational in nature. For example in harmonic maps [N], one is interested in finding $f : M \to N$ with least energy $E(f) = \int_M |df|^2 \, d\,\mathrm{vol}_g$ where (M, g) and (N, h) are smooth Riemannian manifolds with their metrics, $|df|^2$, the energy density (which is the g-trace of the pullback f^*h) that depends on g and h, and f is a C^1 map that is topologically nontrivial, say that it cannot be continuously deformed to a constant map, or that its values are prescribed on the boundary of M. If $M = \mathbf{S}^1$, the circle, then harmonic maps are geodesics (length minimizing curves). If $N = \mathbf{R}$ then f is a harmonic function. If one considers the volume $\mathrm{vol}_N(f(M))$ instead of energy, then the least volume map is a minimal submanifold. We shall consider minimal spheres, $M = \mathbf{S}^2$, in Part II. The volume constrained area minimization problem leads to surfaces with constant mean curvature. Another very important variational problem is the Yamabe Problem, that asks whether the metric of any compact boundaryless manifold (M, g) of dimension m can be conformally deformed to a metric $h = u^{4/(n-2)}g$ of constant scalar curvaure, where $u > 0$ is a function. Yamabe formulated the question variationally: minimize $\int_M |du|^2 + \frac{n-2}{4n-4)} R_g u^2 \, d\,\mathrm{vol}_g$ for $u \in H^1(M)\backslash\{0\}$ and $\int_M |u|^{2n/(n-2)} \, d\,\mathrm{vol}_g = 1$, where R_g is the scalar curvature of a background metric g. The problem was partially solved by Aubin(1976) and completed by Schoen(1984), (see [Au], [SY] for an exposition).

Since the geometric quantities involved, such as length, area, volume, curvature are independent of the choice of coordinate system, the solutions tend to be defined only up to a large group of gauge transformations such as reparameterizations by diffeomorphisms. In the harmonic map problem, the domain is not a subet of Euclidean space, but rather of a differentiable manifold, and the space of competing functions is not a vector space but some nonlinear subspace appropriate for the geometry (e.g., since we may assume $N \subset \mathbf{R}^N$ isometrically for large enough N by the Nash Embedding Theorem, the $\mathbf{R}^N$-vaslued maps take values in N). Thus the problems tend to have an inherently not stricly convex (or nonconvex) nature and analysis proceeds without the benefit of the underlying Euclidean structure.

1. Historical remarks and preliminaries

We shall recall and formulate some basic notions from geometry such as the mean curvature of a surface and describe some of its properties. This material is typically covered in an upper division course on curves and surfaces. Good references are Blaschke & Leichtweiß[BL], Chern[Ch], Courant[Ct], do Carmo[dC], Guggenheimer[Gg], Hicks[Hc], Hopf[Hf], Hsiung[Hs], O'Neill[ON], Oprea[Op1], Struik[Sk]. Good references specializing on minimal and constant mean curvature surfaces are Jost[Jo], Dierkes *et. al.*[DHKW], Lawson[La], Nitsche[Ni], Osserman[O1].

The Plateau Problem. Suppose γ is a closed Jordan curve in Euclidean Space, *i.e.*, a subset homeomorphic to a circle. The Plateau Problem is to find a regular immersed surface with least area having γ as its boundary. It may happen that for some curves, such as one that nearly goes around a circle twice may be spanned by a surface of the type of the disk or the type of a Moebius strip with much less area. There are more complicated curves that bound surfaces with infinitely many topological types, and such that the more complicated the surface, the smaller the area can be made. For that reason, we fix the topological type and try to minimize among parametric surfaces given by maps of a fixed two-manifold with boundary $X : M \to \mathbf{E}^3$. The simplest case is to consider maps from the closed unit disk $\bar{D}$ in the plane. A mapping $X : \bar{D} \to \mathbf{E}^3$ is called piecewise C^1 if it us continuous, and if except along ∂B and along a finite number of regular C^1 arcs and points in the interior D, X is of class C^1. A continuous map $b : \partial B \to \gamma$ is *monotone* if for each $p \in \gamma$, the set $b^{-1}(p)$ is connected. Define a class of maps

$$\mathcal{X}_\gamma = \{X : \bar{D} \to \mathbf{E}^3 : X \text{is piecewise} C^1 \text{ and } X|_\gamma \text{ is a monotone parameterization of } \gamma\}$$

Then we define the area functional $\mathcal{A} : \mathcal{X}_\gamma \to [0, \infty]$ by the following (generally improper) integral.

$$\mathcal{A}(X) = \int_{\bar{D}} \sqrt{\det(g_{ij}(u^1, u^2))}\, du^1\, du^2.$$

Here $(u^1, u^2) \in \bar{D}$ are coordinates in the disk, $X_i = \frac{\partial X}{\partial u^i}$ and $g_{ij} = \langle X_i, X_j \rangle$ where $\langle \cdot, \cdot \rangle$ is the usual Euclidean inner product on $\mathbf{R}^3$. Using the notation $|V|^2 = \langle V, V \rangle$ to denote the square length of a vector, then the integrand can be interpreted as the area of the parallelogram spanned by the X_i's. If $\theta = \angle X_1 X_2$ is the angle, then the squared area of a parallelogram is

$$\begin{aligned} |X_1|^2 |X_2|^2 \sin^2\theta = & \ |X_1|^2|X_2|^2(1 - \cos^2\theta) \\ = & \ |X_1|^2|X_2|^2 \left(1 - \frac{\langle X_1, X_2\rangle^2}{|X_1|^2|X_2|^2}\right) = g_{11}g_{22} - g_{12}^2 = \det(g_{ij}). \end{aligned}$$

In $\mathbf{E}^3$ this also equals $|X_1 \times X_2|^2$. The statement of the problem is to find $X \in \mathcal{X}_\gamma$ so that $\mathcal{A}_\gamma = A(X)$ and

$$\mathcal{A}_\gamma = \inf_{Y \in \mathcal{X}_\gamma} A(Y).$$

The interesting case is if γ satisfies $\mathcal{A}_\gamma < \infty$ which will have to be assumed. There are curves γ for which $\mathcal{A}_\gamma = \infty$. Lawson gives the following example[La]. Imagine starting with a planar circle γ_1. String a number of beads (solid torii) onto γ_1 and replace γ_1 by a new curve γ_2 gotten by splicing in curves coiled a number of times around each bead. Repeat the beading and splicing process for each successive γ_n. Let $\gamma_\infty = \lim_{n\to\infty} \gamma_n$. By estimating the area of the surface needed to span each helical coil, and by selecting the number and dimensions of the beads appropriately, one can arrange that $\mathcal{A}_{\gamma_\infty} = \infty$.

The most significant difficulty in solving the variational problem arises from the fact that the area is independent of parameterization. Thus there is a loss of compactness for minimizers. Douglas found a way to finesse around this difficulty. Thus if $\eta : \bar{D} \to \bar{D}$ is a diffeomorphism then if $X \in \mathcal{X}_\gamma$ then so is $Y = X \circ \eta \in \mathcal{X}_\gamma$ but $A(X) = A(Y)$. This simply follows from the change of variables formula: If $\eta(v^1, v^2) = (u^1, u^2)$ then writing $Y_i = \partial Y / \partial v^i$ and $\tilde{g}_{ij} = \langle Y_i, Y_j \rangle$ then

$$\begin{aligned} Y_i = & \ \frac{\partial Y}{\partial v^i} = \frac{\partial X}{\partial u^k}\frac{\partial u^k}{\partial v^i} \Rightarrow \tilde{g}_{ij} = g_{k\ell}\frac{\partial u^k}{\partial v^i}\frac{\partial u^\ell}{\partial v^j} \Rightarrow \\ \sqrt{\det(\tilde{g}_{ij})}dv^1\, dv^2 = & \ \sqrt{\det(g_{k\ell})}\left|\det\left(\frac{\partial u^k}{\partial v^i}\right)\right|\left|\det\left(\frac{\partial v^i}{\partial u^j}\right)\right| du^1\, du^2 = \sqrt{\det(g_{ij})}du^1\, du^2. \end{aligned}$$

Nonparametric surfaces and the Constant mean Curvature equation. If we assume that $f(u^1,u^2)$ has minimal area among all competitors, then we may derive the Euler equation as follows. Assume that $\zeta \in C_0^2(B)$ is a function with compact support and consider the variation $X[t] = (u^1,u^2, f(u^1,u^2)+t\zeta(u^1,u^2))$. Since $A[0] \le A[t]$ for a minimizer, the first derivative vanishes. Differentiating inside the integral, and integrating by parts,

$$0 = \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(x[t]) = \int_B \frac{f_1 v_1 + f_2 v_2}{\sqrt{1+f_1^2+f_2^2}} du^1 \wedge du^2, = -\int_B \operatorname{div}\left(\frac{(f_1,f_2)}{\sqrt{1+f_1^2+f_2^2}}\right) v\, du^1 \wedge du^2.$$

Since, v is arbitrary, the remaining term must vanish. The resulting divergence structure elliptic equation is the *minimal surface equation.*

$$\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0. \tag{1.1}$$

where $\nabla f = (f_1, f_2)$.

Similarly, if we assume that the volume under the surface is kept constant, then we minimize $\mathcal{A}$ under the condition that

$$\mathcal{V}(X) = \int_B f\, du^1 \wedge du^2 = c,$$

where c is a constant. Besides this equation, the minimizer satisfies the Euler Lagrange equation

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} (\mathcal{A}(x[t]) + \lambda \mathcal{V}(X[t])) \\ &= \int_B \left\{ \frac{\langle \nabla f, \nabla v\rangle}{\sqrt{1+|\nabla f|^2}} + \lambda v \right\} du^1 \wedge du^2, \\ &= \int_B \left\{ -\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) + \lambda \right\} v\, du^1 \wedge du^2. \end{aligned}$$

where λ is the (constant) Lagrange multiplier. The constrained optimizers satisfy the *constant mean curvature equation* (CMC equation.)

$$\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = \lambda. \tag{1.2}$$

However, it may happen that the minimizing or CMC surface for some γ does not project to B as a graph. Instead we consider parametric surfaces given by the vector function $X : \bar{D} \to \mathbf{R}^3$ be a mapping of the closed unit disk which is continuous on the closed domain $\bar{D}$ and twice continuously differentiable on D. We assume that $X(u^1,u^2)$ is regular, which means that the cross product $X_1 \times X_2$ is a nonvanishing vector field on $X(D)$. The area of $X(D)$ is given by

$$A(X) = \int_D |X_1 \times X_2|\, du^1\, du^2.$$

Suppose that $\gamma : \mathbf{S}^1 = \partial D \to \mathbf{R}^3$ is a continuous one-to-one mapping from the unit circle to three space. The *Plateau Problem* is to find $Z \in \mathcal{X} = \{X \in C(\bar{D}, \mathbf{R}^3) \cap C^2(D, \mathbf{R}^3) :\ X \text{ is regular}\}$ which minimizes area among all such maps

$$A(Z) = \inf_{X \in \mathcal{X}} A(X).$$

The second fundamental form and a geometric interpretation of mean curvature. We describe a geometric interpretation of mean curvature, for arbitrary surfaces in Euclidean space. Suppose we're given a parametric surface locally by $X(u^1, u^2)$ near the point p. The tangent plane to $X(M)$ at point $X(u^1, u^2)$ is spanned by the tangent vectors x_1 and X_2 by applying the Gram-Schmidt algorithm to the vector functions, it is possible to find arthonormal vector fields E_1, E_2 that span the tangent space at $X(u^1, u^2)$ and which vary in a C^1 fashion. We can also let $E_3 = E_1 \times E_2$ be the unit vector field normal to the surface. Since the surface is regular, it can be represented as a graph over the tangent plane, so for each p, we may write $X(M)$ as a graph over the tangent plane near p as

$$X(\xi^1, \xi^2) = X(p^1, p^2) + \xi^1 E_1(p^1, p^2) + \xi^2 E_2(p^1, p^2) + f(\xi^1, \xi^2; p^1, p^2) E_3(p^1, p^2).$$

Since E_1 and E_2 are tangent to $X(M)$ at P, $f_1(0,0;p) = f_2(0,0;p) = 0$ (at the point $X(p)$.) The *second fundamental form* is defined to be the Hessian matrix $h_{ij}(p) = f_{ij}(0,0;p)$. The mean curvatrure is the trace $H = \frac{1}{2}(h_{11} + h_{22}) = \frac{1}{2}(\kappa_1 + \kappa_2)$ and the Gauß curvature is the determinant $K = \det(h_{ij}) = \kappa_1 \kappa_2$, where κ_i are the eigenvalues of h_{ij} at p. These numbers are called the principal curvatures. Because H and K are symmetric functions of eigenvalues, they are defined independently of he choice of the orthonormal basis at p. Thus H and K are invariantly defined quantities of the surface. It turns out that one can account for the effect of the nonzero slope and that the expression (1.2) gives the mean curvature with $\lambda = 2H$.

Another way to compute is to use the covariant derivative $\nabla_V X$ of a vector function X in the V direction. This simply means to take the directional derivative and to orthogonally project it to the tangent plane $\bar{\nabla}_V X = \Pi(\bar{\nabla}_V X)$ where $\Pi(\xi^1 E_1 + \xi^2 E_2 + \xi^3 E_3) = \xi^1 E_1 + \xi^2 E_2$. Then $h_{ij} = -\langle \nabla_{E_i} E_3, E_j \rangle = \langle \nabla_{E_i} E_j, E_3 \rangle$. It follows that the quadratic form $\frac{1}{2} h_{ij}(p) \xi^i \xi^j$ is a second order approximation to the surface in the tangent plane, and is sometimes called the *shape operator.*

Complex analysis and isothermal coordinates. The parameter manifold (M^2, ds^2) can be thought of as a Riemannian surface, that is for each local chart there is a symmetric, positive definite $ds^2 = \sum_{i,j=1}^2 g_{ij}(u^1, u^2)\, du^i\, du^j$. It turns out, that by a (local) diffeomorphism, it is possible to find isothermal coordinates (x^1, x^2) in which the metric takes the form $ds^2 = e^{2\varphi}((dx^1)^2 + (dx^2)^2)$.

Theorem. *Suppose M^2 is a surface with boundary, homeomorphic to the unit disk $\bar{D}$ in the plane via the chart $X : \bar{D} \to M$. Suppose the coefficients of the metric tensor of M can be defined in this chart by bounded measurable functions g_{ij} with $\det(g_{ij}) \geq c > 0$ in D. Then M admits a conformal representation $\tau \in H^{1,2} \cap C^\alpha(\bar{B}, \bar{D})$, where B is the unit disk and τ satisfies almost everywhere the conformality relations*

$$|\tau_1|^2 = |\tau_2|^2, \qquad \langle \tau_1, \tau_2 \rangle = 0,$$

where (x^1, x^2) denote the coordinates in $\bar{B}$ and the inner product is given by the metric of M so in terms of g_{ij} on $\bar{D}$. moreover τ can be normalized by the three point condition, namely three prescribed points on the boundary of D can be made to correspond, respectively to three points on the boundary of $\bar{B}$. Furthermore, τ is as regular as M, i.e., if M is of class $C^{k,\alpha}$ ($k \in \mathbf{N}, 0 < \alpha < 1$) or C^∞ then $\tau \in C^{k,\alpha}(\bar{B})$ or $C^\infty(\bar{B})$, resp. In particular, if $k \geq 1$ then the conformality relations are satisfied everywhere and τ is a diffeomorphism.

For a proof of this, see Jost [Jost]. The local version, known as the Korn-Lichtenstein theorem, was proved by Lavrenitiev and Morrey for this generality. Morrey and Jost extended it a global result on multiply connected domains.

If the two-manifold is sufficiently regular then at each point there is a neihborhood in which by a change of coordinates, the metric is given in this way. The Gauss curvature is then

$$K = -e^{-2\varphi} \left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} \right) \varphi.$$

Gauss's *Theorema Egregium* says that for a surface in $\mathbf{R}^3$ with the induced metric from Euclidean space, the curvature computed intrinsically this way using just the metric agrees with the extrinsic computation using the second fundamental form.

One of the most important formulas in elementary differential geometry is the *Gauss-Bonnet formula.* The easiest proof relies on isothermal coordinates on small pieces. Let (M^2, g) be any orientable two dimensional Riemannian manifold which is closed and without boundary. Then

$$\int_M K \, d\,\text{Area} = 2\pi\chi(M), \tag{1.3}$$

where $\chi(M)$ is the Euler characteristic. The Euler characteristic is a topological invariant that may be computed for M as follows: take a triangulation of M into finitely many curvilinear polygons. Then $\chi(M) = b_2 - b_1 + b_0$ where b_2 is the number of faces, b_1 is the number of edges and b_0 is the number of vertices in the triangulation. For example $\chi(\text{sphere}) = 2$, $\chi(\text{torus}) = 0$ and $\chi(\text{two holed torus}) = -2$.

Curvature. For higher dimensional manifolds (M^n, g), the sectional, Ricci and scalar curvatures may be computed from the metric restricted to two dimensional slices of the manifold. First, given a two plane $P \subset T_zM$ in the tangent space at $z \in M$, we describe how to compute the sectional curvature of the two-plane $K(P)$. The exponential map $\exp_z : T_zM \to M$ is defined on rays. For any unit vector $W \in T_zM$, let $t \mapsto \gamma(t) = \exp_z(tW)$ be the unit speed geodesic with initial data $\gamma(0) = z$ and $\gamma'(0) = W$. Thus if $B_\varepsilon(0) \subset T_zM$ is a sufficiently small ball, then $L_{z,P} = \exp_z(P \cap B_\varepsilon(0))$ is a small two dimensional surface in M which is tangent to P at z. Then the sectional curvature $K(P)(z)$ is just the Gauss curvature of the two dimensional manifold $(L_{z,P}, g|_{L_{z,P}})$ at the point z. For example, the sectional curvature of the standard unit n-sphere $\mathbf{S}^n$ is $K(P)(z) = 1$ because $L_{z,P}$ agrees with the great unit 2-sphere through z and tangent to P. For a unit vector $W \in T_zM$, let $\{W, E_2, \ldots, E_n\}$ be an orthonormal basis for T_zM. The Ricci curvature is $\text{Ric}(W,W)(z) = \sum_{j=2}^n K(E_j)(z)$ and $\text{Ric}(V,W)$ is its polarized form. So for $\mathbf{S}^n$, $\text{Ric}(W,W) = n-1$ for all W, z. The scalar curvature $R_g(z) = \sum_{j=1}^n \text{Ric}(E_j, E_j)(z)$ is the sum over an orthonormal basis in T_zM. Thus for $\mathbf{S}^n$, $R_g = n(n-1)$.

2. The reverse isoperimertic inequality under integral bounds on curvature: deformation of elastic rings under hydrostatic pressure

The classical isoperimetric inequality stated for plane curves is one of the first variational problems a student encounters. Let $X = \{\Gamma \in C^2(\mathbf{R}, \mathbf{R}^2) : \Gamma \text{ is } 2\pi \text{ periodic, injective on } [0, 2\pi) \text{ and positively oriented}\}$ be the space of embedded closed curves. Then the length and (signed) area enclosed are given by

$$L(\Gamma) = \int_0^{2\pi} \left|\dot\Gamma(t)\right| \, dt, \qquad \text{Area}(\Gamma) = \int_\Gamma x \, dy.$$

The isoperimetric problem, solved by the circle, is to find the greatest $\text{Area}(\Gamma)$ among curves $\Gamma \in X$ so that $L(\Gamma) \le L_0$ where $L_0 \ge 0$ is a constant. The Euler Lagrange equation for this problem is

$$K = \text{const.}$$

where K is the curvature of the curve. To compute K, change parameter to arclength

$$s = \int_0^t \left|\dot\Gamma(t)\right| \, dt$$

so $T = \frac{d\Gamma}{ds} = (\cos\theta, \sin\theta)$ is the unit tangent vector where $\theta = \angle(\mathbf{e}_1, T)$ is its angle from horizontal. If $\Gamma \in C^2$ then the curvature is $K = \frac{d\theta}{ds} = \left|\frac{dT}{ds}\right|$.

The *reverse isoperimetric problem* is to minimize $\text{Area}(\Gamma)$ for $\Gamma \in X$ so that $L(\Gamma) \ge L_0$. Of course without other conditions, thre is no solution in X and the solutions degenerate to loops enclosing zero area. We shall describe two different additional constraints under which the reverse problem can be solved: the case of integral bounded curvature and the case of pointwise bounded curvature.

The problem with an additional integral constraint. We wish to minimize Area(Γ) for $\Gamma \in X$ so that $L(\Gamma) \geq L_0$ and so that $\int_\Gamma \kappa^2\, ds \leq K_0$, where $K_0 > 0$ is constant. The dual problem, with highest order term, the bending energy $E(\Gamma)$ as the objective function

$$\begin{aligned} \text{Minimize} \quad & E(\Gamma) = \int_\Gamma \kappa^2\, ds, \\ \text{Subject to} \quad & \Gamma \in X, \quad L(\Gamma) \geq L_0 \text{ and } \text{Area}(\Gamma) \leq A_0. \end{aligned}$$

The problem of minimum bending energy for curves (elastica) with fixed endpoints and given length was proposed by J & D. Bernoulli and studied by Euler [Tl]. This problem spurred the development of the calculus of variations and the theory of elliptic functions. Elastica in three space and other spaceforms [LS1], [LS2], as well as dynamical deformations [LS3] have been studied. Elasatica with given turning angle are discussed in [Op1]. Buckling of a circular ring under hydrostatic pressure has een studied by several authors. Carrier [Cr], Chaskalovic and Naili determine bifurcation points [CN]. The buckling and stability of elastic rings is well studied [An], [At], [Ka], [Ko], [Ta], [TO].

The ring problem can be regarded as planar deformation of a cross section of an elastic tube under hydrostatic pressure. This model arose in our study of a design for a nanotube electromechanical pressure sensor [LT], [WZ], [ZT]. Single walled carbon nanotubes were first created in the laboratory over a decade ago [I], [II]. Hydrostatic pressure forces the volume reduction of a nanotube. Its walls essentially keep a fixed cross section length, have area depending on pressure, but resist by minimizing bending energy. The electrical response to a large deformation is a metal to semiconductor transition resulting in a decrease in conductance. Since the amount of deformation for different pressures depends on length, by devising an array of nanotubes of various sizes, any conductance response can be engineered into the sensor.

Let s denote arclength along a curve Γ. The position vector is then $X(s) = (x(s), y(s))$. Since we are parameterizing by arclength, the unit tangent vector is given by

$$T(s) = (x'(s), y'(s)) = (\cos\theta(s), \sin\theta(s)). \tag{2.1}$$

Here prime denotes differentiation with respect to arclength. The position

$$X(s) = X_0 + \int_0^s (\cos\theta(\sigma), \sin\theta(\sigma))\, d\sigma$$

may be recovered by integrating.

The cross section of the tube is to be regarded as an inextensible elastic rod which is subject to a nonstant normal hydrostatic pressure $\mathcal{P}$ along its outer boundary. The rod is assumed to bend in the plane and have a uniform wall thickness h_0 and elastic properties. The centerline of the wall is given by a smooth embedded closed curve in the plane $\Gamma \subset \mathbf{R}^2$ which bound a compact region Ω whose boundary has given length L_0 and which encloses a given area $A(\Omega)$. Among such curves we seek one, Γ_0, that minimizes the energy

$$\mathcal{E}(\Gamma) = \frac{\mathcal{B}}{2} \int_\Gamma (K - K_0)^2\, ds + \mathcal{P}\left(A(\Omega) - A_0\right),$$

where $\mathcal{B} = E h_0^3 / \{12(1-\nu^2)\}$ is the flexural rigidity modulus, E is Young's modulus, ν is Poisson's ratio and K denotes the curvature of the curve and K_0 is the undeformed curvature ($= 2\pi/L_0$ for the circle.)

This is equivalent to the problem of minimizing

$$E(\Gamma) = \int_\Gamma K^2\, ds, \tag{2.2}$$

among curves of fixed length at least L_0 that enclose a fixed area Area$(\Omega) \leq A_0$. We are interested in the relation between the geometry of the minimizer and the values of A_0 and L_0. The problem is invariant under a homothetic scaling of Γ_0. Thus if the curve is scaled to $\tilde{\Gamma}_0 = c\Gamma_0$, its area, length and energy change by

$\tilde{A}_0 = c^2 A_0$, $\tilde{L}_0 = cL_0$ and $\tilde{E} = c^{-1}E$ for $c > 0$. Since the shape of the minimizer is independent of the scaled data, it suffices to find the relation between the *Isoperimetric Ratio,* $\mathcal{I}$, and other dimensionless measures of the shape of Γ_0. The isoperimetric ratio $\mathcal{I} = \frac{4\pi A}{L^2}$ satisfies $0 < \mathcal{I} \le 1$ by the isoperimetric inequality, which says that the area of any figure with fixed boundary length does not exceed the area of a circle with that boundary length. Moreover, the only figure with $\mathcal{I} = 1$ is the circle.

To simplify the embeddedness condition, one argues that the extremals have reflection symmetry in two perpendicular directions. Assuming that the minimizing curve has reflection symmetry in both the x and y-directions, we only need to find θ for $0 \le s \le L$ where $4L \ge L_0$, over a quarter of the curve, and then reflect to get the closed curve. The embeddedness will still have to be checked. We are assuming that Γ is a closed C^1 curve. By rotation and translation, we assume $x(0) = y(0) = 0 = x(L_0) = y(L_0)$. In order for the curve not to have a corner at the endpoints, it is necessary that $\theta(0) = 0$ and $\theta(L_0) = 2\pi$. For $\theta(s)$ the minimizer to be embedded, we'll check that the resulting curve $\gamma = X([0, L])$ remains an embedded and in the first quadrant.

Then the area is bounded by Γ, by Green's theorem is

$$\text{Area}(\Gamma) = \int_\Gamma x\,dy = 4\int_\gamma x\,dy \tag{2.3}$$

because $x\,dy$ is zero along the line segments $(x(L), y(L))$ to $(0, y(L))$ to $(0,0)$ which complete γ to a closed curve. The variational problem is to find a function $\theta : [0, L] \to \mathbf{R}$ such that $\theta(0) = 0$, $\theta(L) = \pi/2$ satisfying $\text{Area}(\theta) \le A_0$ which minimizes (2.3).

Euler Lagrange Equation. Since we are looking to minimize E subject to $\text{Area}(\theta) \le A_0/4$, the Lagrange Multiplier $\lambda = 8\mathcal{P}/\mathcal{B} \ge 0$ is nothing more than scaled pressure such that at the minimum, the variations satisfy $4\,\delta E = -\lambda\,\delta\,\text{Area}$. The corresponding Lagrange Functional is thus

$$\begin{aligned}
\mathcal{L}[\gamma] &= 4\int_\gamma K(s)^2\,ds - \lambda\left\{A_0 - \int_\gamma x\,dy\right\} \\
&= 4\int_0^L \dot\theta(s)^2\,ds - \lambda\left\{A_0 - \int_0^L \int_0^s \cos\theta(\sigma)\,d\sigma\ \sin\theta(s)\,ds\right\}.
\end{aligned}$$

Assuming that the minimizer is the function $\theta(s)$ with $\theta(0) = 0$ and $\theta(L) = \pi/2$, we make a variation $\theta + \epsilon v$ where $v \in C^1([0, L])$ with $v(0) = v(L) = 0$. Then

$$\begin{aligned}
0 = \delta\mathcal{L} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{L} = \\
= 8\int_0^L \dot\theta\dot v\,ds - \lambda\int_0^L \left\{\int_0^s v(\sigma)\sin\theta(\sigma)\,d\sigma\ \sin\theta(s) - \int_0^s \cos\theta(\sigma)\,d\sigma\ \cos\theta(s)v(s)\right\} ds
\end{aligned}$$

Integrating by parts, and reversing the order of integration in the second integral,

$$\delta\mathcal{L} = -8\int_0^L \ddot\theta v\,ds - \lambda\left\{\int_0^L\int_\sigma^L \sin\theta(s)\,ds\,v(\sigma)\sin\theta(\sigma)\,d\sigma\ - \int_0^L\int_0^s \cos\theta(\sigma)\,d\sigma\ \cos\theta(s)v(s)\,ds\right\}.$$

Switching names of the integration variables in the second term yields

$$\delta\mathcal{L} = \int_0^L \left[-8\ddot\theta(s) - \lambda\left\{\int_s^L \sin\theta(\sigma)\,d\sigma\ \sin\theta(s) - \int_0^s \cos\theta(\sigma)\,d\sigma\ \cos\theta(s)\right\}\right] v(s)\,ds.$$

Since $v \in C_0^1([0,L])$ was arbitrary, the minimizer satisfies the integro-differential equation

$$\ddot{\theta}(s) = -\frac{\lambda}{8}\left\{\int_s^L \sin\theta(\sigma)\,d\sigma\ \sin\theta(s) - \int_0^s \cos\theta(\sigma)\,d\sigma\ \cos\theta(s)\right\} \tag{2.4}$$

Thus if $\lambda = 0$ we must have $\theta(s) = \frac{\pi s}{2L}$ and γ is a circle of radius L/π. Thus if $\mathcal{I} < 1$ then $\lambda > 0$. To see the differential equation implied by (2.4), we assume that $\dot{\theta} \neq 0$ and differentiate

$$\begin{aligned}
\theta''' &= \frac{\lambda}{8}\left\{\sin\theta(s)\ \sin\theta(s) + \cos\theta(s)\ \cos\theta(s)\right\} \\
&\quad - \frac{\lambda}{8}\left\{\int_s^L \sin\theta(\sigma)\,d\sigma\ \cos\theta(s) + \int_0^s \cos\theta(\sigma)\,d\sigma\ \sin\theta(s)\right\}\theta'(s) \\
&= \frac{\lambda}{8} - \frac{\lambda}{8}\left\{\int_s^L \sin\theta(\sigma)\,d\sigma\ \cos\theta(s) + \int_0^s \cos\theta(\sigma)\,d\sigma\ \sin\theta(s)\right\}\theta'(s) \\
\theta'''' &= \frac{\lambda}{8}\left\{\sin\theta(s)\ \cos\theta(s) - \cos\theta(s)\ \sin\theta(s)\right\}\theta'(s) + \\
&\quad + \frac{\lambda}{8}\left\{\int_s^L \sin\theta(\sigma)\,d\sigma\ \sin\theta(s) - \int_0^s \cos\theta(\sigma)\,d\sigma\ \cos\theta(s)\right\}(\theta'(s))^2 \\
&\quad - \frac{\lambda}{8}\left\{\int_s^L \sin\theta(\sigma)\,d\sigma\ \cos\theta(s) + \int_0^s \cos\theta(\sigma)\,d\sigma\ \sin\theta(s)\right\}\theta''(s)
\end{aligned}$$

from which we get

$$\theta''''\theta' = -\theta''\,(\theta')^3 + \left[\theta''' - \frac{\lambda}{8}\right]\theta''(s). \tag{2.5}$$

This differential equation may be integrated as follows:

$$\frac{\theta''''\theta' - \theta'''\theta''}{(\theta')^2} = \left[\frac{\theta'''}{\theta'}\right]' = -\theta'\theta'' - \frac{\lambda\theta''}{8(\theta')^2} = \left[-\frac{1}{2}(\theta')^2 + \frac{\lambda}{8\theta'}\right]'$$

so there is a constant c_1 so that

$$\theta''' = c_1\theta' - \frac{1}{2}(\theta')^3 + \frac{\lambda}{8}.$$

In other words, the curvature $K = \theta'$ satisfies

$$K'' = c_1 K + \frac{\lambda}{8} - \frac{1}{2}K^3. \tag{2.6}$$

Multiplying by K' and integrating, we find a first integral. For some constant H,

$$(K')^2 = c_1 K^2 + H + \frac{\lambda K - K^4}{4} = F(K). \tag{2.7}$$

Solution of Euler Lagrange Equation. Since the curve closes, the curvature is a L_0-periodic function which satisfies the nonlinear spring equation (2.6). As we expect that the curvature to continue analytically beyond the endpoints of the quarter curve, and as we assume that the curve have reflection symmetries at the endpoints, the curvature would continue as an even function at the endpoints. In particular we'll have $K'(0) = K'(L) = 0$ as in (2.4). Furthermore, as intuition and numerical experience suggests that the optimal curves be elliptical or peanut shapes, the endpoints of the quarter curve are also the minima and maxima of the curvature around the curve, and that we expect these to be the only critical points of curvature. Since the minimum K may be negative, as in peanut shaped regions, the embeddedness of the reflection is more likely to be satisfied if $K(0) = K_1$ is the maximum of the curvature and $K(L) = K_2$ is the minimum of curvature around the curve.

One degree of freedom in the problem is homothety, which will be irrelevant to deducing nondimensional measures, as we've already remarked. Indeed, if the curve is scaled $\tilde{X} = cX$ then $\tilde{K} = c^{-1}K$, $d\tilde{K}/d\tilde{s} = c^{-2}K'$, $\tilde{c}_1 = c^{-2}c_1$, $\tilde{H} = c^{-4}H$ and $\tilde{\lambda} = c^{-3}\lambda$. For convenience, as $\lambda > 0$ for noncircular regions, we set $\lambda = 1$ to fix the scaling.

As K and K' vary, they satisfy (2.7), thus the parameters c_1, H, λ must allow solvability of (2.7). Moreover, $0 = F(K_1) = F(K_2)$ and the points $(K_1, 0)$ and $(K_2, 0)$ must be in the same component of the solution curve of (2.7) in phase (K, K') space. Thus, given K_1, K_2 we can solve for c_1 and H,

$$c_1 = \frac{1}{4}\left(K_1^2 + K_2^2 - \frac{\lambda}{K_1 + K_2}\right), \tag{2.8}$$

$$H = -\frac{K_1 K_2}{4}\left(K_1 K_2 + \frac{\lambda}{K_1 + K_2}\right), \tag{2.9}$$

provided $K_2 \neq -K_1$. A solution would have a minimum and maximum curvature with appropriate c_1 and H so we assume the solvability condition. Then $4F(K) = Q_1(K)Q_2(K)$ can be factored into quadratic polynomials, where

$$\begin{aligned} Q_1 &= (K_1 - K)(K - K_2); \\ Q_2 &= K^2 + (K_1 + K_2)K + K_1 K_2 + \frac{\lambda}{K_1 + K_2} \end{aligned}$$

Since we've assumed that $F(K)$ is positive in the interval $K_2 < K < K_1$, this forces other inequalities among the c_1, H and λ. For example, if $K_2 = 0$, then $H = 0$ and $Q_2 > 0$ near $K = 0$ only if $\lambda = 1$, which we assume to be true. For $K_2 < 0$, then $Q_2 > 0$ near $K = 0$ for some K_1 only if $K_1 + K_2 > 0$, which we also assume.

Since the possible homotheties and translations of the same solution (shifts like $K(s + c)$) have been eliminated, the remaining indeterminacy coming from the constants of integration is to ensure that the direction angle Θ changes by exactly $\pi/2$ over γ. Thus given K_2, we solve for K_1 so that $\Theta(L) = \pi/2$ where

$$\Theta(L) = \int_0^{L_0} K(s)\, ds = \int_{K_2}^{K_1} \frac{K\, dK}{\sqrt{F(K)}}, \tag{2.10}$$

We have used equation (2.7) to change variables from s to $K(s)$. In fact, this integral can be reduced to a complete elliptic integral. Similarly

$$L = \int_0^{L_0} ds = \int_{K_2}^{K_1} \frac{dK}{\sqrt{F(K)}} \tag{2.11}$$

is a complete elliptic integral. In order to graph closed solutions of (2.6), we choose K_2, then find c_1 and H using (2.8),(2.9). Then find K_1 so that (2.10) holds. Then compute L using (2.11) and integrate (2.2),(2.1),(2.3),(2.6) numerically on $0 \leq s \leq L$. K_1 is found using a simple root finder to solve $\Theta(L) = \pi/2$).

Reduction to Elliptic Integrals. We now describe the reduction of (2.10),(2.11) to complete elliptic integrals, following the procedure [AS], [HH]. Choose a constant μ so that $Q_2 - \mu Q_1$ is a perfect square. This happens upon the vanishing of the discriminant

$$\Delta = D^2(\mu + 1)^2 - 4S^2\mu - 4(\mu + 1)\frac{\lambda}{S}$$

where $S = K_1 + K_2$, $D = K_1 - K_2$ and $P = K_1K_2$. It is zero when μ equals one of

$$\mu_1, \mu_2 = \frac{S^3 + 4PS + 2\lambda \pm \sqrt{(\lambda + 2K_1S^2)(\lambda + 2K_2S^2)}}{SD^2}. \tag{2.12}$$

The factors are

$$\begin{aligned}(1+\mu_1)K^2 + (1-\mu_1)SK + (1+\mu_1)P + \frac{\lambda}{S} = Q_2 - \mu_1 Q_1 &= F_1^2 = (\alpha K - \beta)^2 \\ (1+\mu_2)K^2 + (1-\mu_2)SK + (1+\mu_2)P + \frac{\lambda}{S} = Q_2 - \mu_2 Q_1 &= F_2^2 = (\eta K + \delta)^2.\end{aligned}$$

The signs were chosen based on numerical values. It follows that

$$\begin{aligned}\alpha &= \sqrt{1+\mu_1} \\ \beta &= \sqrt{(1+\mu_1)P + \frac{\lambda}{S}} \\ \eta &= \sqrt{1+\mu_2} \\ \delta &= \sqrt{(1+\mu_2)P + \frac{\lambda}{S}},\end{aligned}$$

which turn out to be positive. We can now solve for the factors as sums of squares

$$\begin{aligned}Q_1 &= \frac{F_1^2 - F_2^2}{\mu_2 - \mu_1}, \\ Q_2 &= \frac{\mu_2 F_1^2 - \mu_1 F_2^2}{\mu_2 - \mu_1}.\end{aligned}$$

The idea is to change variables in the integral according to

$$T = \frac{F_1}{F_2} = \frac{\alpha K - \beta}{\eta K + \delta}, \qquad K = \frac{\beta + \delta T}{\alpha - \eta T}, \qquad \frac{dT}{dK} = \frac{\alpha\delta + \beta\eta}{(\eta K + \delta)^2}.$$

The function T is increasing. Since $Q_1(K_1) = Q_1(K_2) = 0$ it follows that $T = 1$ when $K = K_1$ and $T = -1$ when $K = K_2$. Moreover,

$$Q_1Q_2 = \frac{(F_1^2 - F_2^2)(\mu_2F_1^2 - \mu_1F_2^2)}{(\mu_2 - \mu_1)^2} = \frac{(T^2 - 1)(\mu_2T^2 - \mu_1)F_2^4}{(\mu_2 - \mu_1)^2}$$

Therefore, the integral (2.11) becomes

$$L = \frac{2(\mu_1 - \mu_2)}{(\alpha\delta + \beta\eta)\sqrt{\mu_1}} \int_{-1}^{1} \frac{dT}{\sqrt{(1 - T^2)(1 - \frac{\mu_2}{\mu_1}T^2)}} = \frac{4(\mu_1 - \mu_2)}{(\alpha\delta + \beta\eta)\sqrt{\mu_1}} \mathcal{K}(m) \tag{2.13}$$

where $m = \sqrt{\mu_2/\mu_1}$ is imaginary and

$$\mathcal{K}(m) = \int_0^1 \frac{dT}{\sqrt{(1 - T^2)(1 - m^2T^2)}}$$

is the complete elliptic integral of the first kind.

To find $\Theta(L)$ we express K by partial fractions

$$K = \frac{\beta + \delta T}{\alpha - \eta T} = \frac{(\alpha\delta + \beta\eta)T}{\alpha^2 - \eta^2 T^2} + \frac{\frac{\delta}{\eta} + \frac{\beta}{\alpha}}{1 - \frac{\eta^2}{\alpha^2}T^2} - \frac{\delta}{\eta}$$

Because the first term is odd, we get

$$\begin{aligned} \Theta &= \frac{2(\mu_1 - \mu_2)}{(\alpha\delta + \beta\eta)\sqrt{\mu_1}} \int_{-1}^{1} \frac{K\,dT}{\sqrt{(1-T^2)(1+m^2T^2)}} \\ &= \frac{4(\mu_1 - \mu_2)}{\alpha\eta\sqrt{\mu_1}} \Pi\left(\frac{\eta^2}{\alpha^2}, m\right) - \frac{\delta}{\eta} L \end{aligned} \tag{2.14}$$

where

$$\Pi(n,m) = \int_0^1 \frac{dT}{(1-nT^2)\sqrt{(1-T^2)(1-m^2T^2)}}$$

is the complete elliptic integral of the third kind.

We can also write the solution $K(s)$ in terms of elliptic integrals. Expressing the incomplete integral corresponding to (2.15), we find by substituting $T = -\mathrm{cn}(\nu)$ (see [AS], p. 596) that

$$\begin{aligned} s &= \frac{2(\mu_1 - \mu_2)}{(\alpha\delta + \beta\eta)\sqrt{\mu_1}} \int_{-1}^{T} \frac{dT}{\sqrt{(1-T^2)(1-\frac{\mu_2}{\mu_1}T^2)}} \\ &= \frac{2(\mu_1 - \mu_2)}{(\alpha\delta + \beta\eta)\sqrt{\mu_1 + \mu_2}} \,\mathrm{cn}^{-1}\left(T \,\middle|\, \frac{-\mu_2}{\mu_1 + \mu_2}\right). \end{aligned} \tag{2.15}$$

It follows that

$$T = -\mathrm{cn}\left(\zeta s \,\middle|\, \frac{-\mu_2}{\mu_1 + \mu_2}\right)$$

so that

$$K = \frac{\beta - \delta\,\mathrm{cn}(\zeta s)}{\alpha + \eta\,\mathrm{cn}(\zeta s)}$$

where

$$\zeta = \frac{(\alpha\delta + \beta\eta)\sqrt{\mu_1 + \mu_2}}{4(\mu_1 - \mu_2)}.$$

This is the result of Levy [Lv] and Carrier [Cr]. As a check, at zero this is $K(0) = K_2 = (\beta - \delta)/(\alpha + \eta)$ as it is also a root of Q_1. Similarly at L, where $K(L) = K_1 = (\beta + \delta)/(\alpha - \eta)$.

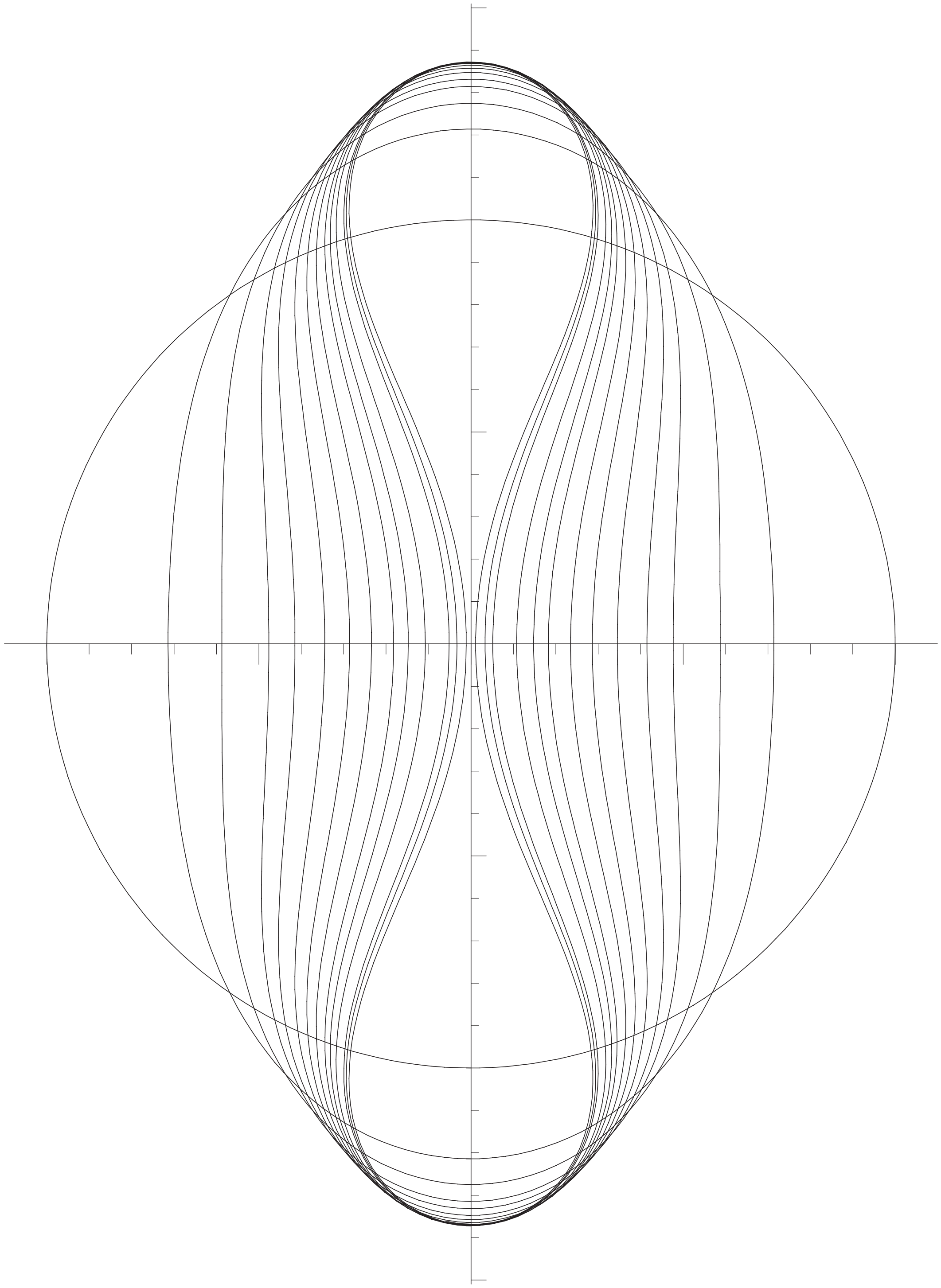

Figure 1.

Graphical results. First we observe that the circle is the limiting figure as $D \to 0$. The formulas (2.12) are not effective for computation for small D, however, the expressions (2.11),(2.14) may be recomputed in terms of $D^2\mu_i$ and become nonsingular as $D \to 0$. To see the limiting circle, make the change of variable

$$K = \frac{S}{2} + \frac{D}{2}T,$$

in equation (2.14) to find

$$\Theta = \int_{-1}^{1} \frac{2(S+DT)\sqrt{S}\,dT}{\sqrt{(1-T^2)(4S^3 - SD^2 + 4\lambda + 4S^2TD + ST^2D^2)}} \to \frac{\pi\sqrt{S^3}}{\sqrt{S^3+\lambda}}$$

as $D \to 0$. Since $\pi/n = \Theta(L)$ it follows that the, limiting $2K_0 = S_0 = \lambda(n^2-1)^{-1/3}$ so the circle has radius $R_0 = 2(n^2-1)^{1/3}/\lambda$. The figures remain embedded for $K_2 > -.2878$, suggesting that the embedded minimizer of the variational problems is not given by these figures for isoperimetric ratios below the critical $\mathcal{I}_0 = .270949$. The ratio $\mathcal{I}_c = .819469$ is the transition point between convex and nonconvex minimizers.

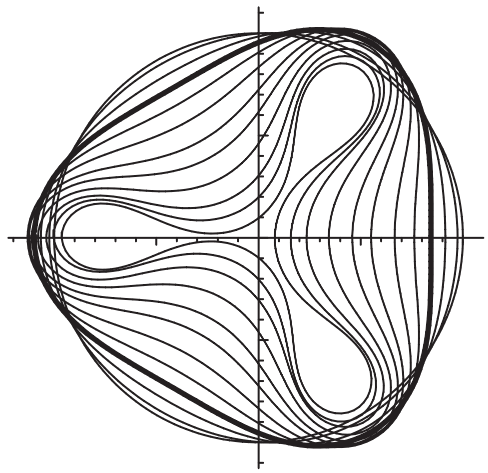

Figure 2.

What evidence is there that the other closed curves that satisfy (2.6) are not the minimizers? One possibility is that K is periodic of period L_0/n where $n \geq 3$. We must have at least $n = 2$ (four critical points of curvature) because of the Four Vertex Theorem for closed plane curves [dC]. For example, there are closed curves with $\Theta(L) = \frac{\pi}{3}$. Then $L_0 = 6L$ and the other variables are suitably increased. The curve $\gamma = \gamma([0, L])$ makes up one sixth of the boundary. The area inside Γ is then six times the area between γ and the y-axis plus the area of the equilateral triangle whose base is $2x(L)$. Thus $A_0 = 6A(L) + \sqrt{3}[x(L)]^2$. This time, the ratio $\mathcal{I}_c = .935405$ is the transition point between convex and nonconvex minimizers and the figures remain embedded for $K_2 > -.516$. The energy is higher for this family of solutions than for the $n = 2$ family. Several examples are plotted in Figure 2.

The Willmore Problem. We indicate an open problem that can be considered as a generalization of the ring problem, in that it concerns a quadratic (second order) curvature integrand. The *Willmore Problem* is to show that among all immersed torii Σ in $\mathbf{R}^3$, the functional

$$\mathcal{W}(\Sigma) = \int_\Sigma H^2 \, d\,\text{Area} \geq 2\pi^2$$

with equality if and only if Σ is the *anchor ring.* An anchor ring is the image under stereographic projection of translates in $\mathbf{S}^3$ and $\mathbf{R}^3$ of the *Clifford torus.* The Clifford torus is the minimal surface in $\mathbf{S}^3$ given by $\mathbf{R}^2 \ni (\theta_1, \theta_2) \mapsto \frac{1}{\sqrt{2}}(\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2) \in \mathbf{S}^3 \subset \mathbf{R}^4$. Stereographic projection $\sigma : \mathbf{S}^3 \to \mathbf{R}^3$ is the conformal map given by $(x, y, z, w) \mapsto \left(\frac{x}{1-w}, \frac{y}{1-w}, \frac{z}{1-w}\right)$. L. Simon has proved that the minimizer of $\mathcal{W}$ among torii exists and is smooth, embedded and unknotted [Sn]. For an account of recent progress, see *e.g.,* [Rs].

3. The reverse isoperimertic inequality under pointwise bounds on curvature

The curvature condition in the reverse isoperimetric problem is now replaced by a pointwise bound

$|K(s)| \le K_0$ for all s , where $K_0 > 0$ is constant.

$$\begin{aligned} &\text{Minimize} && \text{Area}(\Gamma), \\ &\text{Subject to} && \Gamma \in X, \quad \|K\|_\infty \le K_0 \text{ and } L(\Gamma) \ge L_0. \end{aligned}$$

One may imagine a bicycle chain that flexes freely, but up to a limit, as far as its pins allow, which can be modelled by a uniform bound on the curvature. For short chains, the least area is again a peanut shape. We shall only sketch the solution to this problem, the full details may be found in our paper with Howard [HT].

For simplicity sake, let us dilate so that $K_0 = 1$. Since we expect discontinuities in the minimizers, we shall consider the space of embedded curves $X = \{\gamma : \mathbf{S}^1 \to \mathbf{R}^2 : \gamma \in C^{1,1} \text{ and } \gamma(\mathbf{S}^1) \text{ is embedded}\}$. By the Jordan curve theorem, $\gamma \in X$ bounds a topological disk we call $M \subset \mathbf{R}^2$ so that $\gamma = \partial M$. We call curves whose curvature is bounded by $|\kappa| \le 1$ in this weak sense *curves of class* $\mathcal{K}$. When represented by arclength parameter, $\gamma \in \mathcal{K}$ satisfies

$$|\gamma'(s_1) - \gamma'(s_2)| \le |s_1 - s_2| \qquad \text{for all } s_1,\, s_2.$$

Thus γ_s is differentiable *a.e.* and satisfies $\|\gamma_s\|_\infty \le 1$. Some other extremal problems for such curves have been studied previously. For example, the problem of finding the shortest plane curve of class $\mathcal{K}$ with given endpoint and starting line element (position and direction) was solved by Markov, *e.g.* [Pv]. The problem of finding the shortest plane curve of class $\mathcal{K}$ given starting and ending line elements was solved by Dubins [D].

Theorem 3.1. Reverse Isoperimetric Inequality. *If M is an embedded closed disk in the plane $\mathbf{R}^2$ whose boundary curvature satisfies $|\kappa| \le 1$ and with area $A \le \pi + 2\sqrt{3}$ then the length of ∂M is bounded by*

$$\frac{L - 2\pi}{4} \le \text{Arcsin}\left(\frac{A - \pi}{4}\right).$$

If equality holds then M is congruent to a peanut shaped domain as in Figure 3.

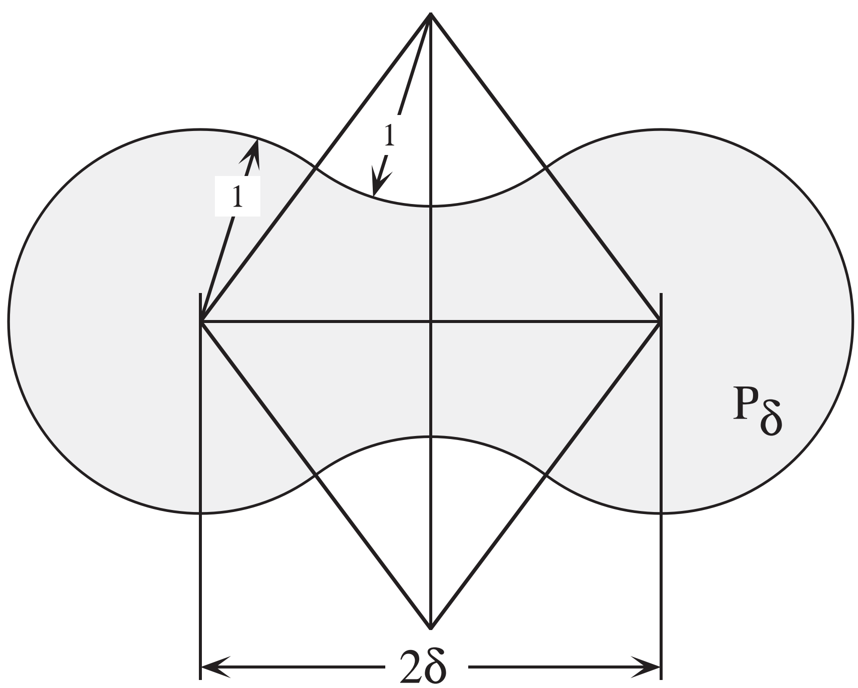

Fig. 3. "Peanut" domain.

There is a threshold phenomenon: if the area is larger than $\pi + 2\sqrt{3}$ then there is no upper bound for the length of ∂M. This is the area of the pinched peanut domain $P_{\sqrt{3}}$. Examples can be found by breaking a thin peanut and connecting the ends with a long narrow strip. In fact, the set of possible points (A, L) for embedded disks whose boundary satisfies $|\kappa| \le 1$ is further restricted.

To prove existence and uniqueness of the extremal figures, we use a replacement argument to show that extremals are piecewise circular arcs. Compactness depends on apriori length bounds. The results depend on a theorem of Pestov and Ionin [PI] on the existence of a large disk in a domain with uniformly bounded curvature (see e.g. [BZ].) We include an argument for Pestov and Ionin's theorem along the lines of Lagunov's [L] proof of the higher dimensional generalization using analysis of the structure of the cut set of such a domain. Lagunov gives a sharp lower bound for the radius of the biggest ball enclosed within hypersurfaces all of whose principal curvatures are bounded $|\kappa_i| \le 1$. Lagunov and Fet [LF] show that the bound is increased if additional topological hypotheses are imposed. It is noteworthy that the examples which show the sharpness of the Lagunov and Lagunov-Fet bounds for dimension greater than one are not unit spheres. Our results use both the existence of a disk and structure of the cut set.

Let $\mathcal{M}(A)$ denote the space of all embedded closed disks $M \subset \mathbf{R}^2$ whose boundary curves are in class $\mathcal{K}$ and whose areas is A. Let $\mathcal{N}(L)$ denote the space of all embedded closed disks $M \subset \mathbf{R}^2$ whose boundary curves are in class $\mathcal{K}$ and whose length $|\partial M| = L$. Then we say $E \in \mathcal{M}(A)$ is extremal if $|\partial E| = \sup\{|\partial M| : M \in \mathcal{M}(A)\}$. Similarly, $E \in \mathcal{N}(L)$ is extremal if $|M| = \inf\{|M| : M \in \mathcal{N}(L)\}$. Although these problems are dual, they require slightly different treatment. By similar analyses, all possibilities of curves in $\mathcal{K}$ may be summarized.

Theorem 3.2. *The set of pairs (A, L) where A is the area and L is the boundary length of $M \subset \mathbf{R}^2$, an embedded closed disk whose boundary is of class $\mathcal{K}$, consists exactly of the points in the first quadrant (shown in Figure 4.) satisfying three inequalities:*

(1) *The isoperimetric inequality*

$$4\pi A \le L^2.$$

Equality holds if and only if M is a circular disk.

(2) *The reverse isoperimetric inequality. If $2\pi \le L < 14\pi/3$ then there holds*

$$\sin\left(\frac{L-2\pi}{4}\right) \le \frac{A-\pi}{4}. \tag{4.1}$$

Equality holds in (4.1) if and only if M is congruent to the peanut P_δ (Figure 1.) where

$$\delta = 4\sin\left(\frac{L-2\pi}{8}\right).$$

(3) *Embeddedness border. If $L \ge 14\pi/3$ then*

$$A > \pi + 2\sqrt{3}.$$

Equality cannot hold, although there are arbitrarily nearby regions for which the embeddedness degenerates by "puckering". For example one can consider a sequence of domains decreasing to the dumbbell region consisting of two unit disks, two triangles with circular sides and a segment of length $L/2 - 7\pi/3$.

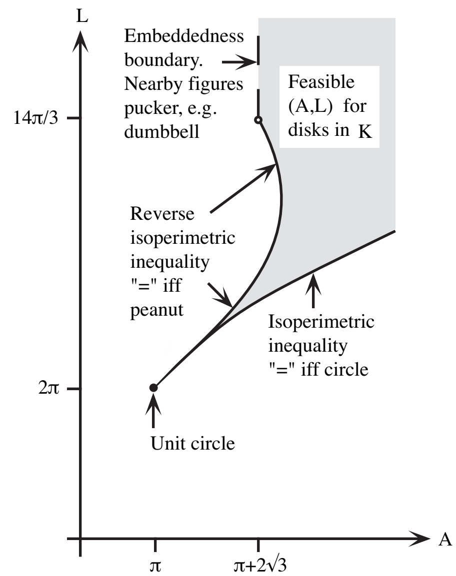

Fig. 4. Feasible region.

We shall give a brief indication of the ideas.

Compactness in the class $\mathcal{K}$ is immediate because the minimizing sequence is bounded in $C^{1,1}$ provided that there is a bound on length. For the length minimization problem this follows from Theorem 3.2. A subsequence converges to a candidate curve with bounded curvature. It remains to show that the extreme curves are peanuts P_δ.

.

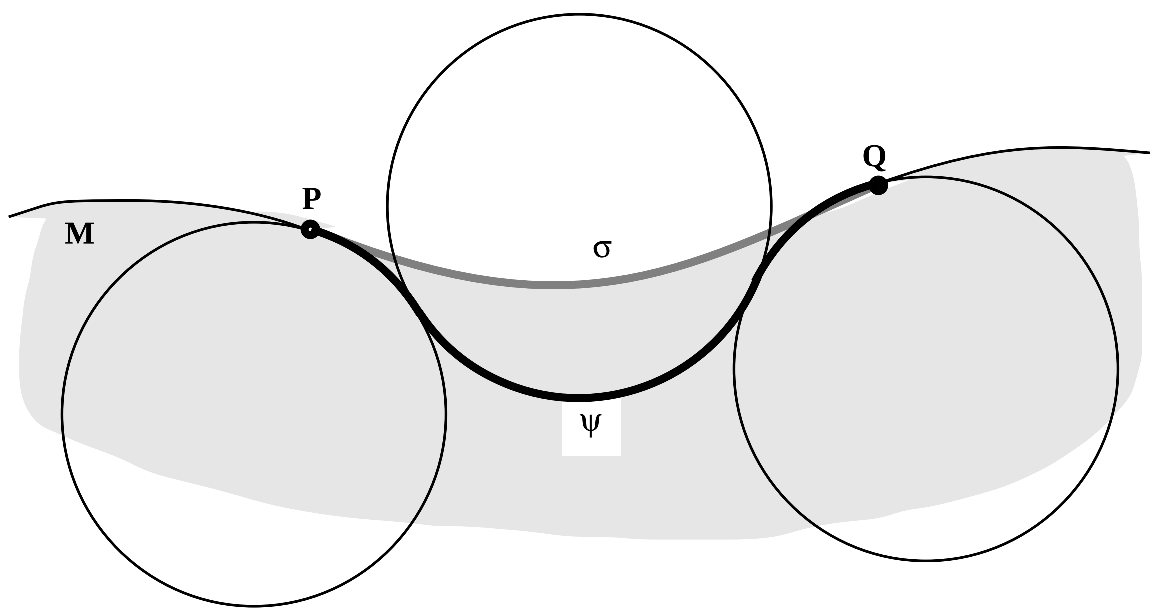

Fig. 5. Replacement argument for a concave arc.

The first step is to show that the extreme curve consists of finitely many arcs of unit circles. If this were not the case, we show that by replacing a short ($L(\sigma) < \pi/3$) segment σ of the curve $\gamma = \partial M$ by another consisting of arcs of unit circles, we can increase the length or decrease the area or both. By outward dilation of the replacement curve we maintain membership in $\mathcal{K}$ because the curvature $|K|$ decreases and satisfy the constraint, but violate the extremality. Actually there are a number of cases that have to be checked. If, for example as in Figure 5., there is a short concave segment $\sigma \subset \partial M$, then a competeing curve ψ consisting of three circular arcs can be constructed by taking arcs from the inside osculating unit circles at the endpoints and splicing in a circular arc to connect the outer arcs. One shows that any embedded arc with the same ending elements (position and direction) as σ must stay outside ψ and thus ψ reduces the area, and be shorter than ψ.

To finish, a similar argument shows that every convex arc must have length at least $\pi/2$. Since the total length is bounded by $14\pi/3$, that limits the number different circular arcs (to 6.) A little calculus is used to show that the peanuts are extremal. By calculating the dimensions of the peanut, we obtain the sharp inequality.

Let us now give an indication if the proof of the preliminary reverse isoperimetric inequality, needed for the compactness argument. Observe that $\pi + 2\sqrt{3} = \text{Area}(P_1)$, the peanut whose outer disks are tangent.

Theorem 3.3. *Suppose $\gamma \in \mathcal{K}$ is a closed curve of bounded curvature. If* $\text{Area}(\gamma) < \pi + 2\sqrt{3}$ *then $L \le 2A$.*

The theorem of Pestov and Ionin and the structure of the cut locus. Pestov and Ionin proved that C^2 disks with $|K| \le 1$ must contain a unit disk. Lagunov and Fet's argument applies to curves in $\mathcal{K}$. Following [HT], the idea is to consider the *cut set* of ∂M. Roughly, the cut set is the set of points in M equally distant from several boundary points. Let M be a simply connected plane domain with C^1 boundary which satisfies a one-sided condition on the curvature. Let the boundary curve of M be positively oriented, parameterized by arclength, γ' absolutely continuous and $\langle \gamma'(s+h) - \gamma'(s)), N(s)\rangle \le h$ for all s and $0 < h < \pi$. Equivalently, the boundary ∂M has curvature satisfying $\kappa_g \le 1$ *a.e.* We denote the class of all such curves by $\mathcal{K}^+$.

Proposition 3.4. *(Pestov and Ionin* [PI]*) Let $M \subset \mathbf{R}^2$ be an embedded disk whose boundary is of class $\mathcal{K}^+$. Then M contains a disk of radius one. In particular the area of M is at least π with equality if and only if M is a disk of radius one.*

Outline of the proof. For $X \in \partial M$ let $C(X)$ be the first point P along the inward normal to ∂M at X where the segment $[X, P(X)]$ stops minimizing $\text{dist}(P, \partial M)$. Call this the cut point of $X \in \partial M$ in M. From the definition it is clear that M contains a disk of radius $\text{dist}(X, C(X))$ about $C(X)$. Lemma 3.5 shows that if $C(X)$ is the cut point of $X \in \partial M$, then at least one of the following two conditions holds

1. $C(X)$ is a focal point of ∂M along the normal line to ∂M at X, or
2. there is at least one other point $Y \in \partial M$ so that $C(Y) = C(X)$ and

$$|C(X) - X| = |C(X) - Y| = \text{dist}(C(X), \partial M).$$

(For example, if the boundary were C^2, see [CE, Lemma 5.2 page 93].) If $C(X)$ is a focal point of ∂M then the curvature condition implies $|X - C(X)| \ge 1$ by Lemma 3.6 and we are done. However, if $\mathcal{C}$ denotes the set of all cut points then we will show that $\mathcal{C}$ contains at least one focal point. □

We elaborate. For any $X \in \partial M$ let $\eta_X(s)$ be the unit speed geodesic, $\eta_X(0) = X$ with $\eta_X{}'(0)$ equal to the inward unit normal to ∂M. The *cut point* of $X \in \partial M$ is the point $\eta_X(s_0)$ where s_0 is the supremum of all $s > 0$ so that the segment $\eta_X([0, s])$ realizes the distance $\text{dist}(\eta_X(s), \partial M)$. The *focal point* of $X \in \partial M$ is the point $\eta_X(s_1)$ where s_1 is supremum of values $s > 0$ so that the function on ∂M defined by $Y \mapsto \text{dist}(\eta_X(s), Y)$ has a local minimum at $Y = X$. If ∂M is C^2 at X then s_1 is the first s where $Y \mapsto \text{dist}(\eta_X(s), Y)$ ceases to have a positive second derivative at $Y = X$. It is possible that no such s_1 exists; in this case we say that the *focal distance* is $s_1 = \infty$. Clearly $s_0 \le s_1$. In geometric optics, the focal points are called the *caustics.*

Denote by $\mathcal{C}$ the set of all cut points of ∂M in M. What is the local geometry of $\mathcal{C}$ like at its "nice" points?

Lemma 3.5. *Any point $P \in \mathcal{C}$ satisfies at least one of the following two conditions*

1. *P is a focal point of ∂M or*
2. *There are two or more distance minimizing geodesics from ∂M to P.*

Proof. This is standard. If $P \in \mathcal{C}$ is not a focal point of ∂M then let $r := \mathrm{dist}(P, \partial M)$ and let $X \in \partial M$ be a point with $P = \eta_X(r)$. Then choose a sequence $s_k \searrow r$ such that for each k there is a point $X_k \in \partial M$ so that $\eta_X(s_k) = \eta_{X_k}(r_k)$ for some $r_k < s_k$. By going to a subsequence we can assume that $X_k \to Y$ for some $Y \in \partial M$. Because P is not focal point of ∂M we have $Y \neq X$. It follows that $\eta_Y(r) = P$ and η_Y is a minimizing geodesic from ∂M to P. □

Lemma 3.6. *Let $M \subset \mathbf{R}^2$ be a domain whose boundary is of class $\mathcal{K}^+$. Let $Y \in \mathcal{C}$ be a focal point. Then* $\mathrm{dist}(Y, \partial M) \geq 1$.

Proof. Let $Y = \eta_X(s_0)$ for some point $X \in \partial M$ and $s_0 > 0$. Let $\gamma \in \mathcal{K}^+$ denote the boundary curve ∂M parameterized so that $\gamma(0) = X$. Since γ is tangent to ∂M at X, by the fact that $\|K\|_\infty \leq 1$, some interval $\gamma((-\varepsilon, \varepsilon))$ is not contained in the open disk $B_s(\eta_X(s))$ for each $0 < s < 1$. Hence $\partial M \ni Z \mapsto \mathrm{dist}(Z, \eta_X(s))$ has a local minimum at $Z = X$. Thus $s_0 \geq 1$. □

Lemma 3.7 (Structure of the cut locus away from focal points.). *Let $P \in \mathcal{C}$ be a cut point that is not a focal point and let $r = \mathrm{dist}(P, \partial M)$. Then there is a finite number of $k \geq 2$ of minimizing geodesics from P to ∂M, and*

Case 1: *If $k = 2$, then there is a neighborhood U of P so that $\mathcal{C} \cap U$ is a C^1 curve and the tangent to $\mathcal{C}$ at P bisects the angle between the two minimizing geodesics from P to ∂M.*

Case 2: *If $k \geq 3$, then the k geodesic segments from P to ∂M split the disk $B_r(P)$ into k sectors $S_1, \ldots, S_k$. There is a small open disk U about P so that in each sector S_i the set $\mathcal{C} \cap U \cap S_i$ is a C^1 curve ending at P and the tangent to this curve at P is the angle bisector of the two sides of the sector S_i at P.*

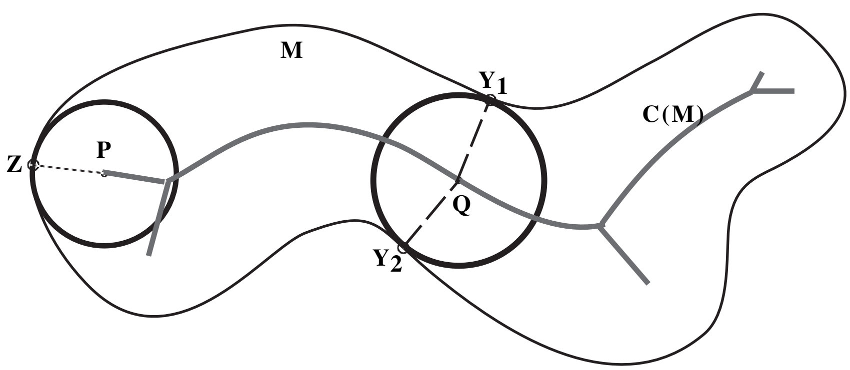

Fig. 6. Cut set. P is a focal point. Q is not.

If there were no focal points then $\mathcal{C}(M)$ would consist of a graph consisting of C^1 curves, meeting at junctions with valence ≥ 3. In particular, there would be no terminating nodes. However, we have assumed that M is topologically the disk. Since the cut set is a deformation retract of M (along normals to the boundary), such a cut set must then be a *tree.* However, every nonempty tree has terminating vertices, which is a contradiction.

Thus M must contain a unit disk. In fact, if you pick a point in the regular part of $\mathcal{C}(M)$, then the same argument shows that there are focal points in $\mathcal{C}(M)$ on both sides of the point.

The next step of the argument is to show that unless ∂M is star shaped with respect to the center point of any of its contained disks, then it must have area greater than $\pi + 2\sqrt{3}$.

First of all, if two disks touch, then M must contain the peanut between the disks. To put it another way, the boundary curve cannot get close to the intersection points of the two disks, This is a maximum principle argument, or in the language of ODE's, there is a field of extremals, $K = -1$ curves, that foliate the triangular region between the disks where no boundary curve can enter.

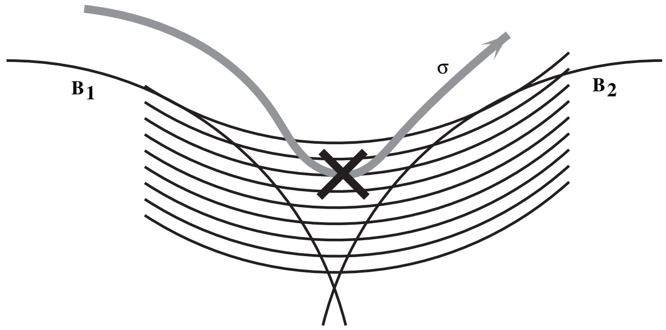

Fig. 7. Field of extremals.

Since the area $\pi + 2\sqrt{3} > \operatorname{Area}(M) \geq \operatorname{Area}(P_\delta) = \pi + 2\delta\sqrt{4-\delta^2}$ it follows that $\delta \leq \varphi(\operatorname{Area} M) < 1$.

If the disks are far enough apart, then a similar argument shows that ∂M avoids triangular fillet regions F near the disks, whose total area exceeds $\operatorname{Area}(P_{\sqrt{3}}) = \pi + 2\sqrt{3}$, so this does not occur. One can also imagine an "earphone shaped" region whose area is large for the same reason. For close but nontouching disks, the argument is that either M contains the spanning peanut or it is earphone shaped so that it must contain a whole unit disk in the complement where the listerners head would go. In both cases M has too much area, so these cases don't occur.

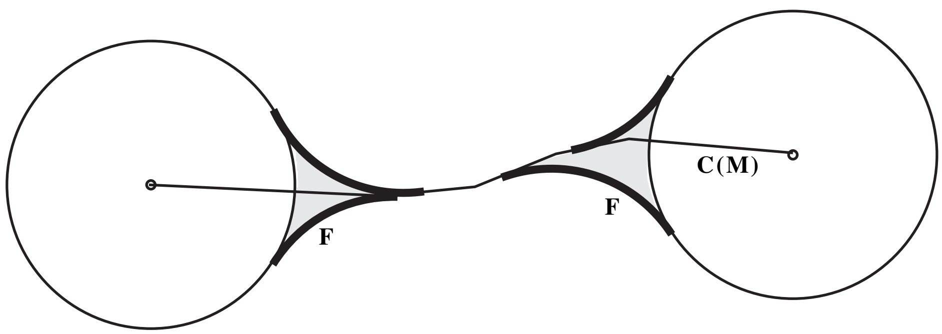

Fig. 8. Configuration with fillets and large area.

The consequence is that if one translates M so that the origin is the center of one of the disks, then $\overline{B_1(0)} \subset M \subset \overline{B_3(0)}$. There is not much maneuvering room: one can show using derivative estimates obtained because the curve turns slower than the circle, that the resulting ∂M is star-shaped with respect to the origin.

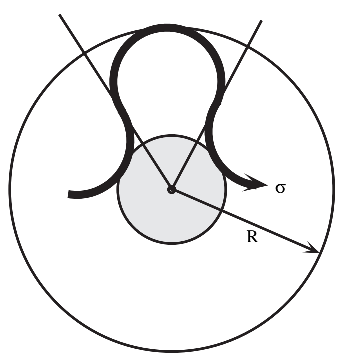

Fig. 9. Small area implies star-shapedness.

Theorem 3.2 is completed if we show that star-shapedness implies the estimate. Formulas with second derivatives are interpreted the weak sense. The result also follows from (in fact gives a derivation of) the Minkowski Formulas.

Lemma 3.8. *Suppose that the curve $\partial M \subset \mathcal{K}$ is star-shaped with respect the origin. Then $L(\partial M) \leq 2\operatorname{Area}(M)$.*

Proof. If we let $\rho(X) = |X|^2$ in $\mathbf{R}^2$ then $\nabla\rho(X) = 2X$. Let $\gamma(s)$ the boundary curve parameterized by arclength, $T = \gamma_s$ be the tangent vector and N be the (inward) unit normal vector which is a $+90°$ rotation of T. Restricting to γ, $\rho = \gamma \cdot \gamma$, $\rho_s = 2\gamma \cdot T$ and $\rho_{ss} = 2 + K\gamma \cdot N$. Star shapedness means that the position vector and inner normal vector satisfy $\gamma \cdot N \le 0$. Integrating on ∂M, using $\|K\|_\infty \le 1$,

$$0 = \tfrac{1}{2}\int_{\partial M} \rho_{ss}\, ds = \int_{\partial M} 1 + K\gamma \cdot N\, ds \ge L(\gamma) + \int_{\partial M} \gamma \cdot N\, ds.$$

On the other hand, integration by parts gives

$$2\,\mathrm{Area}(M) = \tfrac{1}{2}\int_M \Delta\rho\, d\,\mathrm{Area} = -\tfrac{1}{2}\int_{\partial M} \rho_N\, ds = -\int_{\partial M} \gamma \cdot N\, ds$$

and the result follows. On ∂M we have used $\rho_N = N \cdot \nabla\rho = 2\gamma \cdot N$.

A problem of Gromov about pinched curvature. Suppose (M^2, g) is an orientable 2-manifold whose Gauss curvature satisfies $-1 \le K_g < 0$ everywhere on M. By the uniformization theorem, there is a metric g_0 for M so that the Gauss curvature $K_{g_0} = -1$ everywhere on M. Then by the Gauss-Bonnet theorem (1.3),

$$\mathrm{Area}(M, g_0) = -\int_M K_{g_0}\, d\,\mathrm{Area}_{g_0} = -2\pi\chi(M) = -\int_M K_g\, d\,\mathrm{Area}_g \le \int_M d\,\mathrm{Area}_g = \mathrm{Area}(M, g)$$

so that the constant curvature metric minimizes the area of the 2-manifold in this class of metrics. This led Gromov [Gr] to conjecture that for smooth manifolds M^n, $n \ge 3$, that admit metrics whose sectional curvatures satisfy $0 > K_g(P)(z) \ge -1$ for all $z \in M$ and all 2-planes P,

$$\inf_g \mathrm{vol}_n(M, g) \ge \mathrm{vol}_n(M, g_0)$$

if there is a metric with $K_{g_0}(P)(z) = -1$ for all z and P. This problem is undoubtedly intractible by variational means. For other problems and background, see [P].

References

[AS] M. Abramowitz & I. A. Stegun, eds., *Handbook of Mathematical Functions*, National Bureau of Standards, U. S. Government Printing Office, Washington D. C., 1964, republ. Dover, New York, 1965, p. 600..

[An] S. Antmann, *Nonlinear Problems in Elasticity*, in Series: Applied Mathematical Sciences **107**, Springer-Verlag, New York, 1995, pp. 101–116.

[At] T. Atanackovic, *Stability Theory for Elastic Rods*, Series on Stability, Vibration and Control of Systems, Vol. **1**,, World Scientific Publishing Co., Pte. Ltd., Singapore, 1997.

[Au] T. Aubin, *Nonlinear Analysis on Manifolds. Monge-Ampère Equations,*, grundlehren , Vol. **252**,, Springer-Verlag, new York, 1982.

[BL] W. Blaschke & K. Leichtweiß, *Elementare Differentialgeometrie*, Grundlehren, vol. 1, Springer, Berlin, 1973, pp. 72–77.

[BR] W. Blaschke & K. Reidemeister, *Vorlesungen Über Differentialgeometrie.* II. *Affine Differentialgeometrie*, Springer, Berlin, 1923; Reprinted in *Differentialgeometrie* I *&* II, Chelsea, New York, 1967, pp. 57–60.

[Bo] G. Bol, *Isoperimetrisches Ungleighung für Berieche auf Flächen*, Jahresbericht der Deut. Math. Ver. **51** (1941), 219-257.

[BZ] J. D. Burago & V. A. Zalgaller, *Geometric inequalities*, "Nauka", Leningrad, 1980 (Russian); English transl., Grundlehren, vol. 285, Springer, Berlin, 1980, p. 231.

[Cr] G. Carrier, *On the buckling of elastic rings*, Journal of Mathematics and Physics **26** (1947), 94–103.

[CN] J. Chaskalovic & S. Naili, *Bifurcation theory applied to buckling states of a cylindrical shell*, Zeitschrift angewandte Mathematik & Physik (ZAMP) **46** (1995), 149–155.

[CE] J. Cheeger & D. Ebin, *Comparison theorems in Riemannian geometry*, North Holland, Amsterdam, 1975.

[C] S. S. Chern, *Curves and surfaces in Euclidean space*, Studies in Mathematics, Global differential geometry (S. S. Chern, ed.), vol. 27, Mathematical Association of America, Providence, 1989, pp. 99–139.

[Ct] R. Courant, *Dirichlet's Principle, Conformal Mappings and Minimal Surfaces*, Interscience Publ., Inc., New York, 1950.

[DHKW] U. Dierkes, S. Hildebrandt, A. Küster & O. Wohlrab, *Minimal Surfaces* II, Grundlehren der Math. Wiss., vol. 296, Springer, Berlin, 1992, pp. 250–292.

[dC] M. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Englewood Cliffs, 1976, p. 37, p. 406.

[D] L. Dubins, *Curves with minimal length with constraint on average curvature, and with prescribed initial and terminal tangents*, American J. Math. **79** (1957), 497–516.

[EP] C. H. Edwards & D. Penney, *Elementary Differential equations with boundary value problems, 5th ed.*, Pearson / Prentice-Hall, Upper Saddle River, 2004, pp. 517–520.

[F] A. T. Fomenko, *The Plateau Problem, Part I, Historical Survey & Part II, The Present State of the Theory*, Gordon Breach Science Publishers, New York, NY, 1990, Studies in the Development of Modern Mathematics, **1**.

[Gr] M. Gromov, *Volume and bounded cohomology*, Publ. Math. I. H. E. S. **56** (1983), 213–307.

[Gg] H. Guggenheimer, *Differential geometry*, McGraw-Hill, 1963; Reprinted Dover, New York, 1977, 30–31, 229–231..

[G] R. Gulliver, *Regularity of minimizing surfaces of prescribed mean curvature*, Annals of Math. **97** (1973), 275–305.

[GL] R. Gulliver & F. Lesley, *On the boundary branch points of minimal surfaces*, Arch. Rat. Mech. Anal. **52** (1973), 20–25.

[H] H. Hadwiger, *Die erweiterten Steinerschen Formeln für ebene und sphärische Bereiche*, Comment. Math. Helv. **18** (1945/46), 59–72.

[HH] H. Hancock, *Lectures on the Theory of Elliptic Functions*, J. Wiley & Sons Scientific Publications, Stanhope Press, 1909; republ. Dover Publications Inc., New York, 1958, pp. 180–187.

[Hc] N. Hicks, *Notes on Differential Geometry*, van Norstrand Reinhold, Co., New York, NY, 1965, pp. 128–130.

[Hf] H. Hopf, *Differential Geometry in the Large*, Springer Verlag, Berlin, 1983, lecture Notes in Mathematics, **1000**.

[HT] R. Howard & A. Treibergs, *A reverse isoperimetric inequality, stability and extremal theorems for plane curves with bounded curvature*, Rocky Mountain Journal of Mathematics **25** (1995), 635–683.

[Hs] C. C. Hsiung, *A First Course in Differential Geometry*, International Press, Cambridge, MA, 1997, pp. 212–213.

[HRSV] H. Huck, R. Riotzsch, U. Simon, W. Vortisch, R. Walden, B. Wegner & W. Wedland, *beweismethoden der differentiualgeometrie im Großen*, Springer Verlag, Berlin, 1973, lecture Notes in Mathematics, **335**.

[I] S. Iijima,, *Helitic microstructures of graphitic carbon,*, Nature, **354** (1991), 56–58.

[II] S. Iijima & T. Ichihashi, ibid. **363** (1993), 603–605.

[Jo] J. Jost, *Harmonic Maps Between Surfaces*, Springer-Verlag, Berlin, 1984, Lecture Notes in mathematics, **1062**.

[Ka] G. Kämmel, *Der Einfluß der Längsdehnung auf die elastische Stabilität geschlossener Kreisringe*, Acta Mechanika **4** (1967), 34–42.

[K] H. Karcher, *Riemannian comparison constructions*, Studies in Mathematics, Global differential geometry (S. S. Chern, ed.), vol. 27, Mathematical Association of America, Providence, 1989, pp. 170–222.

[Ko] U. Kosel, *Biegelinie eines elasticshen Ringes als Beispiel einer Verzweigungslösung,*, Zeitschrift fur angewandte Mathematik und Mechanik (ZAMM) **64** (1984), 316–319.

[L] V. N. Lagunov, *On the largest possible sphere contained in a given closed surface* I, Siberian Math. Jour. **1** (1960), 205–236; II **2** (1961), 874–880; *(summary)*, Doklad. Akad. Nauk SSSR **127** (1959), 1167–1169. (Russian)

[LF] V. N. Lagunov & I. A. Fet, *Extremal problems for surfaces of prescribed topological type* I, Siberian Math. J. **4** (1963), 145–167; II **6** (1965), 1026–1038. (Russian)

[LS1] J. Langer & D. Singer, *The Total Squared Curvature of Closed Curves*, Jour. Differential Geom. **20** (1984), 1–22.

[LS2] J. Langer & D. Singer, *Knotted Elastic Curves in* $\mathbf{R}^3$, Jour. London Math. Soc.(2) **30** (1984), 512–520.

[LS3] J. Langer & D. Singer, *Curve Straightening and a Minimax Argument for Closed Elastic Curves*, Topology **24** (1985), 75–88.

[La] H. B. Lawson, *Lectures on Minimal Surfaces*, Monografias de Matemática, Instituto de Matemática Pura E Aplicada (IMPA) Consellio Nacional de pesquisas, Rio de Janeiro, 1973.

[Le] J. Lee, *Riemannian Manifolds: An Introduction to Curvature*, Graduate Texts in Mathematics, **176**, Springer, New York, 1991.

[Lv] M. Levy, *Memoire sur un nouveau cas intégrable du probléme de l'elastique et l'une de ses applications*, Journal de Mathématiques Pures et Apliquées, Ser. 3 **7** (1884).

[LT] F. Liu & A. Treibergs, *Lecture on Pressure Deformation of Cross Sections of Nanotubes with Minimal Bending Energy*, Preprint 2003.

[N] S. Nishikawa, *Variational Problems in Geometry*, American Mathematical Society, Providence, 2002, Translations of Mathematical Monographs **205**. Originally published by Iwanami Shoten, Publishers, Tokyo, 1998.

[Ni] J. C. C. Nitsche, *Vorlesungen über Minimalflächen*, Springer-Verlag, Berlin, 1975, Grundlehren der math. Wissenschaften in Einzeldarsetellungen, Bd. **199**.

[ON] B. O'Neill, *Elementary Differential Geometry*, Academic Press, Inc., San Diego, CA., 1966.

[Op1] J. Oprea, *Differential Geometry and its Applications*, Prentice Hall, Upper Saddle River, NJ., 1997, pp. 115–119..

[Op2] J. Oprea, *The Mathematics of Soap Films: Explorations with Maple*©, Student Mathematical Library 10, American Mathematical Society, Providence, 2000, pp. 147–148.

[O1] R. Osserman, *A Survey of Minimal Surfaces* (1969), van Norstrand Reinhold Co., New York, NY., republ. by Dover, Mineola, NY., 1992..

[O2] R. Osserman, *A proof of the regularity everywhere of the classical solution to Plateau's problem*, Annals of Math. **91** (1970), 550–569.

[PI] G. Pestov & V. Ionin, *On the largest possible circle embedded in a given closed curve*, Doklady Akad. Nauk SSSR **127** (1959), 1170–1172. (Russian)

[Pn] P. Petersen, *Comparison geometry problems list*, Riemannian Geometry: Fields Institute Monographs **4**, Amer. Math. Soc., Providence, 1996, pp. 87–115.

[Pv] J. Petrov, *Variational methods in optimal control theory*, Energija, Moscow-Leningerad, 1965 (Russian); English transl., Academic Press, New York, 1968, pp. 124–125.

[Rs] A. Ros, *The isoperimetric and Willmore Prblems*, Contemporary Mathematics **288**: Global Differential Geometry—The Mathematical Legacy of Alfred Gray (M. Fernández & J. Wolf, eds.), American Mathematical Society, Providence, 2000, pp. 149–161.

[St] R. Schmidt, *Critical study of postbuckling analyses of uniformly compressed rings*, Zeitschrift fur angewandte Mathematik und Mechanik (ZAMM) **59** (1979), 581–582.

[SY] R. Schoen & S.-T. Yau, *Lectures in Differential Geometry I*, International Press Inc., Boston, 1994.

[Sn] L. Simon, *Existence of surfaces minimizing the Willmore Functional*, Commun. Anal. Geom. **1** (1993), 281–326.

[Sk] D. Struik, *Lectures on Classical Differential Geometry, 2nd. ed.*, Addison Weseley Publ. Co., Inc., Reading, MA., 1950, republ. by Dover, Mineola, NY., 1988.

[Se] M. Struwe, *Plateau's Problem and the Calculus of Variations*, Princeton Univ. Press, Princeton, NJ., 1988, Mathematical Notes **35**..

[Ta] I. Tadjbakhsh, *Buckled states of elastic rings*, Bifurcation Theory and Nonlinear Eigenvalue Problems, J. Keller & S. Antmann, eds., W. A. Benjamin, Inc., New York, 1969, pp. 69–92, Appendix by S. Antmann, *ibid.* 93–98.

[TO] I. Tadjbakhsh & F. Odeh, *Equilibrium States of Elastic Rings*, J. Math. Anal. Appl. **18** (1967), 59–74.

[Tl] C. Truesdell, *The influence of elasticity on analysis: the classical herirtage*, Bulletin American Math. Soc. **9** (1983), 293–310.

[We] H. Wente, *Counter-example to the Hopf conjecture*, Pacific J. Math. **121** (1986), 193–224.

[WZ] J. Wu, J. Zang, B. Larade, H. Guo, X.G. Gong & Feng Liu, *Computational Designing of Carbon Nanotube Electromechanical Pressure Sensors* (2003), in preparation.

[ZT] J. Zang, F. Liu, Y. Han, A. Treibergs, *Geometric Constant Defining Shape Transitions of Carbon Nanotubes under Pressure*, Physical Review Letters **92** (2004), 105501.1-4.

University of Utah,
Department of Mathematics
155 South 1400 East, Rm 233
Salt Lake City, UTAH 84112-0090
E-mail address: treiberg@math.utah.edu