CHAPTER 18

Vector Calculus

In this chapter we develop the fundamental theorem of the Calculus in two and three dimensions. This begins with a slight reinterpretation of that theorem. Consider the endpoints a, b of the interval $[a, b]$ from a to b as the boundary of that interval. Then the fundamental theorem, in this form:

$$f(b) - f(a) = \int_a^b \frac{df}{dx}(x)dx \ , \tag{18.1}$$

relates the values of a function at the boundary with the values of its derivative in the interior. Stated this way, the fundamental theorems of the Vector Calculus (Green's, Stokes' and Gauss' theorems) are higher dimensional versions of the same idea. However, in higher dimensions, things are far more complex: regions in the plane have curves as boundaries, and for regions in space, the boundary is a surface, and surfaces in space have curves as boundaries. This requires a reinterpretation of the term $f(b) - f(a)$, as a *signed* sum of the values of f on the boundary, the sign being determined by the side on which the interval lies (it is to the right of a and to the left of b). This leads to the understanding that in higher dimensions both sides will be integrals; for example, for a region R in the plane with C as its boundary, the term $f(b) - f(a)$ becomes an integral over the curve C. And in three dimensions, we will have two versions of the fundamental theorem, one relating integrals over a region with integrals over the bounding surface, and another relating integrals over surfaces with integrals over the bounding curve (and with the relation involving some form of differentiation).

We will not give derivations, or even intuitive arguments for the proofs of these theorems. First of all, the idea of the proof is to reduce the theorem to the one-variable fundamental theorem; in this process, the notational complexity is constantly threatening to get out of hand. The proofs then become masterful displays of technical control, and provide little insight. The insight comes from the physical interpretation of these theorems (indeed, so also did the first proofs), particularly in terms of fluid flows. For example, Gauss' theorem simply says that, for a fluid in flow we can measure the rate of change of the amount of fluid in a given region in two ways: directly over the region, or instead, by measuring the rate of passage through the boundary.

§18.1. Vector Fields

A *vector field* is an association of a vector to each point $\mathbf{X}$ of a region R:

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{I} + Q(x,y,z)\mathbf{J} + R(x,y,z)\mathbf{K}\ . \tag{18.2}$$

For example, the vector field

$$\mathbf{X}(x,y,z) = x\mathbf{I} + y\mathbf{J} + z\mathbf{K} \tag{18.3}$$

is the field of vectors pointing outward from the origin, whose length is equal to the distance from the origin. The field $\mathbf{U} = (1/r)\mathbf{X}$ (where $r(x,y,z) = (x^2+y^2+z^2)^{1/2}$) is the unit vector field with the same direction.

Example 18.1 (Gravitation). According to Newton's Law of gravitation, two bodies attract each other with a force proportional to the product of the masses, and inversely proportional to the square of the distance between them. Suppose one body, of mass M is situated at the origin. Then another body of mass m, situated at the point $\mathbf{X}$ experiences the gravitational force due to M:

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{U}\ , \tag{18.4}$$

where G is Newton's universal constant of gravitation, and $\mathbf{U}$ is the unit vector pointing the direction of $\mathbf{X}$. If we want to concentrate on the effect of the mass M on bodies in its vicinity, we introduce the **gravitational field** of M:

$$\mathbf{G}(\mathbf{X}) = -\frac{GM}{r^2}\mathbf{U} = -\frac{GM}{r^3}\mathbf{X}\ . \tag{18.5}$$

Since $\mathbf{F} = m\mathbf{A}$, a body of mass m at $\mathbf{X}$ accelerates toward the origin with acceleration $\mathbf{G}(\mathbf{X})$.

Definition 18.1 *Suppose the region R is filled with a fluid which is in motion. We can describe the motion by following the individual particles. Let $\mathbf{X}(\mathbf{X_0},t)$ be the position at time t of the particle which was at $\mathbf{X_0}$ at time $t=0$. The* **velocity field** *of the motion is the velocity of the particle at position $\mathbf{X}$ at time t, represented by $\mathbf{V}(\mathbf{X},t)$ .This is a time-dependent vector field in the region R. We say that the flow is* **steady** *if its velocity field is independent of time.*

In studying a fluid in motion, we are not interested in the history of particular particles, but in the fluid as a whole. Thus, it is the velocity field of the fluid that is the object of study, rather than the equations of motion. It can be shown that the velocity field completely determines the motion.

Example 18.2 Suppose a fluid is flowing on the plane radially away from the origin. In this case the origin is called a **source**; if the fluid were flowing toward the origin, we call it a **sink**. The equation of motion is given by

$$\mathbf{X}(\mathbf{X_0},t) = f(t)\mathbf{X_0} \quad \text{for some scalar function } f \text{ with } f(0) = 1\ . \tag{18.6}$$

Let's look at the case $f(t) = e^{at}$. We find the velocity field as follows. First, the velocity of the particle originally at $\mathbf{X_0}$ is

$$\frac{\partial}{\partial t}\mathbf{X}(\mathbf{X_0},t) = \frac{d}{dt}(e^{at})\mathbf{X_0} = ae^{at}\mathbf{X_0}\ . \tag{18.7}$$

But this is $a\mathbf{X}$, so the velocity field is $\mathbf{V}(\mathbf{X}) = a\mathbf{X}$, and the flow is steady. However, if, say $f(t) = 1+t$ so that $\mathbf{X}(\mathbf{X_0},t) = (1+t)\mathbf{X_0}$,we have

$$\frac{\partial}{\partial t}\mathbf{X}(\mathbf{X_0},t) = \mathbf{X_0} = (1+t)^{-1}\mathbf{X}\ , \tag{18.8}$$

so the flow is time-dependent.

The terminology may seem confusing: in the first case, the particle's speed is increasing exponentially, while in the second case the particle's speed is constant. But, if we look at a particular point $\mathbf{X}$ in space, then in the first case, the fluid is always moving with the same velocity through that point, while in the second case, the fluid slows down at that point over time.

Example 18.3 Suppose a fluid is rotating on the plane about the origin in the counterclockwise direction at constant angular velocity ω. From the description, this is a steady flow; let's find its velocity field. At a point $\mathbf{X}$, particles move through $\mathbf{X}$ along the circle of radius $|\mathbf{X}|$ at angular velocity ω.Thus the velocity of the fluid at $\mathbf{X}$ is of magnitude $\omega|\mathbf{X}|$ and in the direction tangent to to the circle through $\mathbf{X}$, so $\mathbf{V}(\mathbf{X}) = \omega\mathbf{X}^{\perp}$.

Definition 18.2 *A differentiable function $w = f(x,y,z)$ has associated to it its* **gradient field**

$$\nabla w = \frac{\partial f}{\partial x}\mathbf{I} + \frac{\partial f}{\partial y}\mathbf{J} + \frac{\partial f}{\partial z}\mathbf{K}\ . \tag{18.9}$$

The surfaces $f(x,y,z)$ =const. are orthogonal to the vector field (18.9), and are called the **equipotentials**, *and the function f, a* **potential** *for the field.*

So, the flow associated to a gradient field is easily visualized as being in the direction perpendicular to these equipotential surfaces. A natural question is: when is a vector field $\mathbf{F}$ the gradient of a function; that is, when does a vector field have a potential function? If the vector field with the components $\mathbf{F} = P\mathbf{I} + Q\mathbf{J} + R\mathbf{K}$ is a gradient, so looks like (18.9), then, because of the equality of mixed derivatives, we must have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}\ . \tag{18.10}$$

If these conditions are satisfied, then we can try to find the potential function by integrating one variable at a time.

Example 18.4 Let $\mathbf{F} = (2xy+x)\mathbf{I} + x^2 - y\mathbf{J}$. Is $\mathbf{F}$ a gradient field? If so, find the potential function.

First, we check that the condition (18.10) is satisfied:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy+x) = 2x \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial y}(x^2-y) = 2x\ . \tag{18.11}$$

So, we have a chance of finding a function f such that $\nabla f = \mathbf{F}$. To find f we have to solve the equations

$$\frac{\partial f}{\partial x} = 2xy + x,\ \frac{\partial f}{\partial y} = x^2 - y\ . \tag{18.12}$$

We can find a function satisfying the first equation by integrating with respect to x; so we try $f(x,y) = x^2y - x^2/2$. Now we see if this f satisfies the second equation:

$$\frac{\partial f}{\partial y} = x^2\ , \tag{18.13}$$

which unfortunately is not $x^2 - y$. However, since the derivative with respect to x of any function of y is zero, we could also have tried

$$f(x,y) = x^2y + x^2/2 + \phi(y) \tag{18.14}$$

for some yet-to-be-determined $\phi(y)$. Now, we have, instead of (18.13),

$$\frac{\partial f}{\partial y} = x^2 + \phi'(y) \; ; \tag{18.15}$$

setting that equal to Q gives the equation $\phi'(y) = -y$, so we can take $\phi(y) = -y^2/2$. We conclude that our solution is

$$f(x,y) = x^2y + \frac{x^2}{2} - \frac{y^2}{2} + C \; , \tag{18.16}$$

for any constant C. The reason that the terms involving x disappear in equation (18.13) is precisely that the condition $\partial P/\partial y = \partial Q/\partial x$ is satisfied; if it were not, this procedure would break down at this point.

Example 18.5 The procedure in three dimensions is the same, but longer. Suppose we are given the vector field $\mathbf{F} = (y^2z+1)\mathbf{I} + (2xyz+z)\mathbf{J} + (xy^2+y+1)\mathbf{K}$, and we are told that it is the differential of a function f. Find f.

Since we are told that there is a potential function, we need not verify conditions (18.10). We start with

$$\frac{\partial f}{\partial x} = y^2z + 1 \; . \tag{18.17}$$

Integrating both sides with respect to x, (thinking of y and z as constants), we obtain

$$f(x,y,z) = xy^2z + x + \phi(y,z) \tag{18.18}$$

where ϕ is an unknown function of y and z alone. Now, differentiating this equation, since $\partial f/\partial y = 2xyz + z$, we obtain

$$2xyz + z = 2xyz + \frac{\partial \phi}{\partial y} \; , \tag{18.19}$$

or

$$\frac{\partial \phi}{\partial y} = z \; . \tag{18.20}$$

Now we do the same, integrating both sides with respect to y:

$$\phi(y,z) = yz + \psi(z) \; , \tag{18.21}$$

for some unknown function $\psi(z)$. Thus (18.18) now becomes

$$f(x,y,z) = xy^2z + x + yz + \psi(y,z) \; . \tag{18.22}$$

Differentiating now with respect to z:

$$xy^2 + y + 1 = xy^2 + y + \frac{\partial \psi}{\partial z} \tag{18.23}$$

so $\partial\psi/\partial z = 1$, and thus $\psi(z) = z + C$. Putting this back in (18.22), we have found

$$f(x,y,z) = xy^2z + x + yz + z + C\,. \tag{18.24}$$

The reason that the variable x disappeared from (18.19) and x and y from (18.23) is precisely because of the conditions (18.10); if they did not hold there would be no such function f, and we could not have solved equations (18.20) and (18.23).

Example 18.6 We point out at this time that these methods make sense only in the domain in which the solution function f is well-defined, even if the given vector field is well-defined in a bigger region. Take, for example, the polar function

$$\theta = \arctan\frac{y}{x}\,. \tag{18.25}$$

Since θ is periodic, it is only well-defined (single-valued) in the plane outside of a ray from the origin, say the ray $x \geq 0$. However,

$$\nabla\theta = -\frac{y}{x^2+y^2}\mathbf{I} + \frac{x}{x^2+y^2}\mathbf{J}\,, \tag{18.26}$$

and this is well-defined in the whole plane, except for the origin. Thus, if we apply the above procedure to the vector field (18.26), we get (18.25), and we have to pick a particular branch of the arc tangent.

Two important concepts associated to a vector fields are its **divergence** and **curl**.

Definition 18.3 *Let* **F** *be a vector field given by*

$$\mathbf{F} = P\mathbf{I} + Q\mathbf{J} + R\mathbf{K}\,, \tag{18.27}$$

where P,Q,R are scalar functions. The **divergence** *of* **F** *is*

$$\text{div}\,\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\,, \tag{18.28}$$

and the **curl** *of* **F** *is*

$$\text{curl}\,\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{I} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{J} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{K}\,. \tag{18.29}$$

These are best interpreted in terms of the velocity field of a fluid flow. The divergence is the rate of expansion of the fluid at a point. The curl is a vector describing the rotation of the fluid near the point (the direction of the curl is the axis of rotation and the magnitude is a measure of the rate of rotation). The flow is called **incompressible** if its divergence is zero, and **irrotational** if its curl is zero. We note that the condition (18.10) for a vector field to be a gradient can be expressed as follows:

Proposition 18.1 *Given a differentiable function f, its gradient field is irrotational; that is:* $\text{curl}\,\nabla f = 0$. *In order for a vector field to be a gradient field, it must be irrotational.*

There is a notation which is very convenient in representing the gradient, div and curl. We consider ∇ as an operator on functions:

$$\nabla = \frac{\partial}{\partial x}\mathbf{I} + \frac{\partial}{\partial y}\mathbf{J} + \frac{\partial}{\partial z}\mathbf{K}\,. \tag{18.30}$$

Then, we have

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F}\ , \qquad \text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}\ . \tag{18.31}$$

Two useful formulas are:

$\nabla \cdot (\nabla \times \mathbf{F}) = 0$, or div $(\text{curl}\,\mathbf{F}) = 0$.

$\nabla \times \nabla f = 0$, or curl $(\nabla f) = 0$.

If we are discussing vector fields in two dimensions, we have, for

$$\mathbf{F} = P(x,y)\mathbf{I} + Q(x,y)\mathbf{J}\ , \tag{18.32}$$

$$\text{div}\,\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\ , \tag{18.33}$$

$$\text{curl}\,\mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{K}\ . \tag{18.34}$$

Example 18.7 Find the divergence and curl of the velocity fields a) associated to a source (see example 18.2), and for rotation about a point (see example 18.3).

In example 2 we had $\mathbf{V} = a\mathbf{X} = a(x\mathbf{I} + y\mathbf{J})$. Then

$$\text{div}\,\mathbf{V} = 2a\ , \quad \text{curl}\,\mathbf{V} = 0\ . \tag{18.35}$$

Note that in this case $\mathbf{V} = \nabla r^2/2$, so the field has the circles centered at the origin as equipotentials. In example 3, $\mathbf{V} = \omega\mathbf{X}^{\perp} = \omega(-y\mathbf{I} + x\mathbf{J})$), so that

$$\text{div}\,\mathbf{V} = 0\ , \quad \text{curl}\,\mathbf{V} = 2\omega\mathbf{K}\ , \tag{18.36}$$

and the vector field is not a gradient.

§18.2. Line Integrals and Work

Suppose **F** is a vector field defined on a region R, and C is a curve lying in R. We define the line integral of **F** along C, by analogy with other integrals as follows.

Definition 18.4 *Let $\mathbf{X}_i, 0 \le i \le n$ be a sequence of points on the curve, with $\mathbf{X}_0, \mathbf{X}_n$ the endpoints. Form the sum*

$$\sum_{i=1}^{n} \mathbf{F}(\mathbf{X}_i) \cdot (\mathbf{X}_i - \mathbf{X}_{i-1})\ . \tag{18.37}$$

If the limit of this sum exists (as the maximum distance between successive points approaches zero), it is the **line integral** *of* **F** *along C:*

$$\int_C \mathbf{F} \cdot d\mathbf{X} = \lim_{\max|\Delta\mathbf{X}_i| \to 0} \sum_{i=1}^{n} \mathbf{F}(\mathbf{X}_i) \cdot \Delta\mathbf{X}_i\ , \tag{18.38}$$

where $\Delta\mathbf{X}_i$ represents the vector increment between successive points.

If we have a parametric representation of the curve: $\mathbf{X}(t) = x(t)\mathbf{I} + y(t)\mathbf{J} + z(t)\mathbf{K}$, for $a \le t \le b$, where the functions $x(t), y(t), z(t)$ are differentiable, then we can compute the line integral by integration with respect to t. For, as successive points become arbitrarily close, we can replace each $\Delta\mathbf{X}_i$ by its linear approximation, and in the limit, we obtain

$$\lim \sum_{i=1}^{n} \mathbf{F}(\mathbf{X}_i) \cdot \Delta\mathbf{X} = \lim_{\Delta t_i \to 0} \sum_{i=1}^{n} \mathbf{F}(\mathbf{X}(t_i)) \cdot \frac{d\mathbf{X}}{dt}(t_i)\Delta t_i = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{X}}{dt} dt \ . \tag{18.39}$$

Proposition 18.2 *If C is a curve parametrized by $\mathbf{X} = \mathbf{X}(t)$ for $a \le t \le b$, and $\mathbf{F}$ is a vector field defined on C, then*

$$\int_C \mathbf{F} \cdot d\mathbf{X} = \int_a^b \mathbf{F}(\mathbf{X}(t)) \cdot \frac{d\mathbf{X}}{dt} dt \ . \tag{18.40}$$

Example 18.8 Find $\int_C \mathbf{F} \cdot d\mathbf{X}$ where C is the curve $\mathbf{X}(t) = t^2\mathbf{I} + (t+1)\mathbf{J}$, $0 \le t \le 3$, and $\mathbf{F}(x,y) = x^2\mathbf{I} + xy\mathbf{J}$.

Here

$$\frac{d\mathbf{X}}{dt} = 2t\mathbf{I} + \mathbf{J} \tag{18.41}$$

and, along C,

$$\mathbf{F}(x,y) = x^2\mathbf{I} + xy\mathbf{J} = (t^2)^2\mathbf{I} + t^2(t+1)\mathbf{J} \tag{18.42}$$

so

$$\int_C \mathbf{F} \cdot d\mathbf{X} = \int_0^3 \mathbf{F} \cdot \frac{d\mathbf{X}}{dt} dt = \int_0^3 ((t^2)^2(2t) + t^2(t+1))dt \tag{18.43}$$

$$= \int_0^3 \left(2t^5 + t^3 + t\right) dt = \left(\frac{t^6}{3} + \frac{t^4}{4} + \frac{t^2}{2}\right)_0^3 = 9\left(27 + \frac{9}{4} + \frac{1}{2}\right) = 267.75 \ . \tag{18.44}$$

To summarize, line integrals are computed this way. Let $\mathbf{F} = P\mathbf{I} + Q\mathbf{J} + R\mathbf{K}$ be a vector field in three dimensions, and suppose that C is given parametrically by the equation $\mathbf{X}(t) = x(t)\mathbf{I} + y(t)\mathbf{J} + z(t)\mathbf{K}$, for $a \le t \le b$, where the functions $x(t), y(t), z(t)$ are differentiable. Then

$$\int_C \mathbf{F} \cdot d\mathbf{X} = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{X}}{dt} dt = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right) dt \ . \tag{18.45}$$

If the curve is given as the graph $y = y(x)$, $z = z(x)$, then we still use the same formula, thinking of the parameter as x and the trajectory given by $\mathbf{X}(x) = x\mathbf{I} + y(x)\mathbf{J} + z(x)\mathbf{K}$. Of course, as we have defined the line integral, it is independent of the parametrization of the curve, and depends only on the direction along the curve in which we integrate.

The line integral (18.45) may appear in several different forms. First, if we want to interpret the line integral as the integral of a differential (as in all cases of integration), we write (18.45) as

(18.46) $$\int_a^b Pdx + Qdy + Rdz \ ,$$

as the integral of the differential $Pdx + Qdy + Rdz$. To calculate the integral, we choose a convenient parametrization and calculate as in (18.45). it is also useful to refer to the parametrization by arc length. Since $d\mathbf{X}/ds = \mathbf{T}$ where $\mathbf{T}$ is the unit tangent to the curve. we can write $d\mathbf{X} = \mathbf{T}ds$ and the line integral is

(18.47) $$\int_C \mathbf{F}\cdot d\mathbf{X} = \int_C \mathbf{F}\cdot \mathbf{T}ds \ .$$

This expresses the line integral as the integral with respect to arc length of the component of the field in the direction of the curve. Finally we note that the integral is additive over curves.

Proposition 18.3 *If the curve C can be written as a finite succession of curves $C_1,\dots,C_n$ such that the initial point of each C_i is the same as the terminal point of its predecessor, then, for any vector fied* $\mathbf{F}$ *defined on C:*

(18.48) $$\int_C \mathbf{F}\cdot d\mathbf{X} = \int_{C_1} \mathbf{F}\cdot d\mathbf{X} + \cdots + \int_{C_n} \mathbf{F}\cdot d\mathbf{X} \ .$$

Example 18.9 Find $\int_C \mathbf{F}\cdot d\mathbf{X}$, where $\mathbf{F}(x,y) = xy\mathbf{I} + y^2\mathbf{J}$, and C is the triangle from (0,0) to (2,0) to (3,0) and back to (0,0).

C consists of three line segments:

(18.49) $$C_1 : 0 \le x \le 2,\ y = 0 \quad C_2 : 0 \le y \le 3,\ x = 2 - \frac{2}{3}y \quad C_3 : 3 \ge y \ge 0,\ x = 0 \ .$$

We calculate the three integrals separately, and then, by (18.48), take their sum. On C_1, we take x as the parameter, and $dy = 0$.

(18.50) $$\int_{C_1} \mathbf{F}\cdot d\mathbf{X} = \int_{C_1} xydx + y^2dy = \int_0^2 0dx = 0 \ .$$

On C_2 we take y as the parameter, and we have $dx = -(2/3)dy$.

(18.51) $$\int_{C_2} \mathbf{F}\cdot d\mathbf{X} = \int_{C_2} xydx + y^2dy = \int_0^3 (2 - \frac{2}{3}y)(-\frac{2}{3})dy + y^2dy$$

(18.52) $$= \int_0^3 \left(-\frac{4}{3} + \frac{4}{3}y + y^2\right) dy = \left(-\frac{4}{3}y + \frac{2}{3}y^2 + \frac{y^3}{3}\right)\Big|_0^3 = 11 \ .$$

Finally, since $x = 0$ on C_3:

(18.53) $$\int_{C_3} \mathbf{F}\cdot d\mathbf{X} = \int_{C_3} y^2dy = \int_3^0 y^2dy = -9 \ ,$$

and

$$\int_C \mathbf{F}\cdot d\mathbf{X} = \int_{C_1}\mathbf{F}\cdot d\mathbf{X} + \int_{C_2}\mathbf{F}\cdot d\mathbf{X} + \int_{C_3}\mathbf{F}\cdot d\mathbf{X} = 0 + 11 - 9 = 2\ . \tag{18.54}$$

Example 18.10 Find $\int_C \mathbf{F}\cdot d\mathbf{X}$, where $\mathbf{F}(x,y) = y\mathbf{I} + x\mathbf{J}$, and C is the curve given parametrically as $x = 1 + 3\cos t,\ y = 3\sin(2t)$.

We first calculate the differentials $dx = -3\sin t dt,\ dy = 6\cos(2t)dt$, so

$$\mathbf{F}\cdot d\mathbf{X} = 3\sin(2t)(-3\sin t dt) + (1+3\cos t)(6\cos(2t)dt) \tag{18.55}$$

$$= (-9\sin(2t)\sin t + 6\cos(2t) + 18\cos(2t)\sin(t))dt\ . \tag{18.56}$$

Performing the integration, we get $\int_C \mathbf{F}\cdot d\mathbf{X} = -.00111$.

If $\mathbf{F}$ is a force field in the plane in space, then the work done in moving from one point $\mathbf{X}_0$ to another point $\mathbf{X}_1$ is $W = \mathbf{F}\cdot(\mathbf{X}_1 - \mathbf{X}_0)$, since the action of the force is only in the direction from $\mathbf{X}_0$ to $\mathbf{X}_1$. Now, if $\mathbf{X}(t)$ represents a curve C then the contribution to work along a small piece of the curve $d\mathbf{X}$ is $dW = F\cdot d\mathbf{X}$. We find the **total work** done by the force along the trajectory as the integral:

$$Work = \int_C F\cdot d\mathbf{X}\ . \tag{18.57}$$

Example 18.11. Let $\mathbf{F} = -z\mathbf{I} + x\mathbf{J} + \mathbf{K}$ be a force field in space. How much work is done by this force in moving an object from the origin to the point (1,1,1) along the path $C:\ y = x^2,\ z = x^3$?

First we express C parametrically by $\mathbf{X} = x\mathbf{I} + x^2\mathbf{J} + x^3\mathbf{K},\ 0 \le x \le 1$, so that $d\mathbf{X}/dx = \mathbf{I} + 2x\mathbf{J} + 3x^2\mathbf{K}$. The force along C is, in terms of the parameter x: $\mathbf{F} = -x^3\mathbf{I} + x\mathbf{J} + \mathbf{K}$. Then, the work done by this force is

$$\int_C \mathbf{F}\cdot d\mathbf{X} = \int_0^1 (-x^3 + 2x(x) + 3x^2)dx = \int_0^1 (5x^2 - x^3)dx = \frac{17}{12}\ . \tag{18.58}$$

Recall that the kinetic energy of a particle of mass m in motion is $(1/2)m|\mathbf{V}|^2$, where $\mathbf{V}$ is its velocity. If we differentiate this with respect to t, and use Newton's Second law $\mathbf{F} = m\mathbf{A}$, we have:

$$\frac{d}{dt}\left(\frac{1}{2}m\mathbf{V}\cdot\mathbf{V}\right) = m\mathbf{A}\cdot\mathbf{V} = \mathbf{F}\cdot\frac{d\mathbf{X}}{dt}\ . \tag{18.59}$$

This expresses the law of conservation of energy for a particle in motion in the presence of a force field: the change in the kinetic energy along the trajectory is equal to the work done to the particle. For suppose that the particle travels along the path C from time $t = a$ to $t = b$. We integrate (18.40) along the path, getting

$$\frac{m}{2}|\mathbf{V}(b)|^2 - \frac{m}{2}|\mathbf{V}(a)|^2 = \int_C \mathbf{F}\cdot d\mathbf{X}\ . \tag{18.60}$$

Example 18.12. A particle of mass 2 g. moves around the circle of radius 1 on the plane in the presence of a centripetal force field (keeping it on the circle) and of the force field $\mathbf{F}(x,y) = (1+y)\mathbf{I} + y^2\mathbf{J}$ (where the magnitude is in newtons). Suppose that at time $t = 0$ the particle is at the point (1,0) travelling at a speed of 3 cm/sec. What is its speed the next time it passes through the point (1,0)?

We parametrize the path using polar coordinates $C: \ x=\cos\theta,\ y=\sin\theta$ for $0\le\theta\le 2\pi$. In terms of this parametrization,

$$\mathbf{F}=(1+\sin\theta)\mathbf{I}+(\sin\theta)\mathbf{J}\ , \quad \frac{d\mathbf{X}}{d\theta}=-\sin\theta\mathbf{I}+\cos\theta\mathbf{J}\ . \tag{18.61}$$

Since the centripetal force is orthogonal to $d\mathbf{X}/d\theta$, the work done in this motion is

$$\int_C \mathbf{F}\cdot d\mathbf{X}=\int_0^{2\pi}(-\sin\theta+\sin^2\theta+\sin\theta\cos\theta)d\theta=\pi\ . \tag{18.62}$$

Since $m=2$, letting b be the time the particle next passes through (1,0), (18.45) gives us

$$\frac{1}{2}|\mathbf{V}(b)|^2=\pi+\frac{1}{2}(3)^2=7.6515\ , \tag{18.63}$$

so $|\mathbf{V}(b)|=3.909\ \text{cm}/\text{sec}$.

§18.3. Independence of Path

In this and the next section, we shall restrict attention to two dimensions. First, let us summarize the preceding sections. A vector field defined in a region D is of the form $\mathbf{F}=P\mathbf{I}+Q\mathbf{J}$ where P and Q are scalar functions on R. If C is a curve in R parametrized by $\mathbf{X}(t)=x(t)\mathbf{I}+y(t)\mathbf{J}$, $a\le t\le b$, then

$$\int_C \mathbf{F}\cdot\mathbf{X}=\int \mathbf{F}\cdot\mathbf{T}ds=\int_C Pdx+Qdy=\int_a^b\left(P\frac{dx}{dt}+Q\frac{dy}{dt}\right)dt\ . \tag{18.64}$$

This is the integral with respect to arc length of the component of $\mathbf{F}$ in the direction of the curve. If $\mathbf{F}$ is a force field, this is the work done by the force along the curve C. If $\mathbf{F}$ is interpreted as the velocity field of a flow, this is the total flow of fluid in the direction of the curve.

We might also be interested in the flow of the fluid across the curve; this is the integral of the component of $\mathbf{F}$ orthogonal to the curve; that is, $\int\mathbf{F}\cdot\mathbf{N}ds$ where $\mathbf{N}$ is the normal to the curve. Since there are two unit normals to the curve, we must specify the direction in which the curve is crossed. For this discussion we shall take the normal pointing to the right of the direction in which the curve is traversed. Since $\mathbf{T}ds=dx\mathbf{I}+dy\mathbf{J}$, we are taking $\mathbf{N}ds=dy\mathbf{I}-dx\mathbf{J}$, so that $\mathbf{F}\cdot\mathbf{N}ds=\det(\mathbf{F},d\mathbf{X})$.

Definition 18.5 *Let $\mathbf{F}=P\mathbf{I}+Q\mathbf{J}$ be a vector field defined in a region R, and C a curve in R. The* **circulation** *of $\mathbf{F}$ along C is*

$$\int\mathbf{F}\cdot\mathbf{T}ds=\int_C\mathbf{F}\cdot d\mathbf{X}=\int_C Pdx+Qdy\ . \tag{18.65}$$

The **flux** *of $\mathbf{F}$ across C from left to right is*

$$\int\mathbf{F}\cdot\mathbf{N}ds=\int_C\det(\mathbf{F},d\mathbf{X})=\int_C -Qdx+Pdy\ . \tag{18.66}$$

Example 18.11 Calculate the circulation and flux of $\mathbf{F}=x^2\mathbf{I}-xy\mathbf{J}$ across the line from (0,0) to (3,4).

The line is easily parametrized by $x = 3t,\ y = 4t,\ 0 \le t \le 1$, so that $dx = 3dt,\ dy = 4dt$. Then

$$\text{(18.67)}\qquad Circulation = \int_C x^2dx - xydy = \int_0^1 (3t)^2(3dt) - (3t)(4t)(4dt) = \int_0^1 (27-48)t^2dt = -7\ .$$

$$\text{(18.68)}\qquad Flux = \int_C xydx + x^2dy = \int_0^1 (3t)(4t)(3dt) - (3t)^2(4dt) = \int_0^1 (36+36)t^2dt = 24\ .$$

Proposition 18.4 *If the vector field* $\mathbf{F}$ *is the gradient of a function in R, then, for any path C,*

$$\text{(18.69)}\qquad \int_C \mathbf{F}\cdot d\mathbf{X} = f(\mathbf{X_1}) - f(\mathbf{X_0})$$

where $\mathbf{X_0}$ *is the initial point of the path, and* $\mathbf{X_1}$ *is its endpoint.*

To see this, let C have the parametrization $\mathbf{X}(t) = x(t)\mathbf{I} + y(t)\mathbf{J}$ for $a \le t \le b$, so that $\mathbf{X_0} = \mathbf{X}(a)$ and $\mathbf{X_1} = \mathbf{X}(b)$. We have

$$\text{(18.70)}\qquad \mathbf{F} = \frac{\partial f}{\partial x}\mathbf{I} + \frac{\partial f}{\partial y}\mathbf{J}$$

so that

$$\text{(18.71)}\qquad \int_C \mathbf{F}\cdot d\mathbf{X} = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right)dt = \int_a^b \frac{d}{dt}f(\mathbf{X}(t))dt = f(\mathbf{X}(b)) - f(\mathbf{X}(a))\ ,$$

by the fundamental theorem of the Calculus.

Definition 18.6 *A region D is called* **connected** *if, for any two points P and Q in D, there is a curve C with endpoints P and Q. A differential* $Pdx + Qdy + Rdz$ *is said to be* **independent of path** *in D if the integral* $\int_C Pdx + Qdy + Rdz$ *is the same for all curves C with the same endpoints. A differential is said to be* **exact** *if it is the differential of a function; that is, there is a function f such that* $df = Pdx + Qdy + Rdz$. *A vector field* $\mathbf{F}$ *is called* **conservative** *if* $\int_C \mathbf{F}\cdot d\mathbf{X}$ *is independent of path.*

Proposition 18.5 *A differential form* $Pdx + Qdy + Rdz$ *defined on a connected region D is independent of path there if and only if it is exact. Equivalently, given a vector field* $\mathbf{F}$*, the line integral* $\int_C \mathbf{F}\cdot d\mathbf{X}$ *is independent of path if and only if* $\mathbf{F} = \nabla f$ *for some function f (called its potential).*

T above proposition tells us that gradient fields are independent of path. Now, we must show that if the differential form $Pdx + Qdy$ is independent of path in D, then it is a gradient. Fix a point (x_0,y_0) in D, and define the function f by $f(x,y) = \int_C Pdx + Qdy$ where C is any path joining (x_0,y_0) to (x,y). To show that $\partial f/\partial x = P$, we take a point $(x+h,y)$ near (x,y), and consider the path C' which is C followed by the line segment L from (x,y) to $(x+h,y)$ (see the figure). Then

$$\text{(18.72)}\qquad f(x+h,y) = \int_{C'} Pdx + Qdy = \int_C Pdx + Qdy + \int_L Pdx + Qdy = f(x,y) + \int_L Pdx + Qdy\ .$$

Now, we can parametrize L by $(x(t),y(t)) = (x+t,y),\ 0 \le t \le h$. Since $dy = 0$ along L, we have

$$\text{(18.73)}\qquad \frac{f(x+h,y) - f(x,y)}{h} = \frac{1}{h}\int_0^h P(x+t,y)dx\ ,$$

which converges to $P(x,y)$. Similarly, $\partial f/\partial y = Q$.

Definition 18.7 *A curve C is said to be* **closed** *if its endpoints are the same (under any parametrization). The integral over a closed curve is denoted* $\oint_C$.

Proposition 18.5 can be restated this way: we have $\mathbf{F} = \nabla f$ if and only if the line integral $\oint_C \mathbf{F}\cdot d\mathbf{X}$ over every closed curve is zero.

Example 18.12 Let $\mathbf{F} = -y\mathbf{I} + x\mathbf{J}$ and C be the boundary of the ellipse $x = 2\cos t,\ y = \sin t,\ 0 \le t \le 2\pi$. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{X} = \oint_C -ydx + xdy = \int_0^{2\pi} -\sin t(-2\sin t)dt + 2\cos t\cos t dt = \tag{18.74}$$

$$2\int_0^{2\pi} (\sin^2 t + \cos^2 t)dt = 4\pi\ . \tag{18.75}$$

§18.4. Green's Theorem in the Plane

Suppose that D is a region in the plane whose boundary is a curve, which we will always consider to be directed so that D always lies to the left of its boundary. We use the notation ∂D to represent the boundary of D so directed. To put it another way: for $\mathbf{T}$ and $\mathbf{N}$ the unit tangent and normal to C as defined in the preceding section, $\mathbf{N}$ is to the right of $\mathbf{T}$, so points out of D. For this reason $\mathbf{N}$ is called the exterior normal.The boundary of a domain is a closed curve (or several closed curves). From the discussion in the preceding section, we know that if $\mathbf{F}$ is a gradient field defined on D, then $\oint_{\partial D} \mathbf{F}\cdot d\mathbf{X} = 0$ and curl $\mathbf{F} = 0$. The connection between these two statements is much deeper and is embodied in Green's theorem which relates the line integral on ∂D with the double integral of curl $\mathbf{F}$ on the domain D. First we state the theorem in differential form.

Proposition 18.6 *(Green's Theorem) Let D be a region, whose boundary* ∂D *is oriented so that D lies to the left of* ∂D*. Suppose that* $Pdx + Qdy$ *is a differential defined on the region D. Then*

$$\oint_{\partial D} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA\ . \tag{18.76}$$

Example 18.13 Let's redo example 14 using Green's theorem, where E represents the region bounded by the ellipse:

$$\oint_C \mathbf{F}\cdot d\mathbf{X} = \oint_C -ydx + xdy = \int\int_E (1+1)dxdy = 4\pi\ , \tag{18.77}$$

since the area of E is 2π.

Example 18.14 Given the differential $x^2dx - xdy$, and D be the rectangle $1 \le x \le 3,\ 1 \le y \le 4$, we have

$$\oint_{\partial D} x^2dx - xdy = \int\int_D (-1+2x)dxdy = \int_1^3\int_1^4 (-1+2x)dydx = 18\ . \tag{18.78}$$

We now restate Green's theorem in two ways in vector form.

Proposition 18.7 *(Stokes' Theorem in the Plane). Let D be a region with boundary ∂D. Let* **F** *be a vector field defined on D. Then*

$$\oint_{\partial D} \mathbf{F}\cdot d\mathbf{X} = \oint_{\partial D} \mathbf{F}\cdot\mathbf{T}ds = \int\int_D \text{curl } \mathbf{F}\cdot\mathbf{K}dA\ . \tag{18.79}$$

This follows directly from (18.76), for if we write $\mathbf{F} = P\mathbf{I} + Q\mathbf{J}$ in component form, we have $\mathbf{F}\cdot d\mathbf{X} = Pdx + Qdy$ and curl $\mathbf{F}\cdot\mathbf{K} = \partial Q/\partial x - \partial P/\partial y$. In terms of fluid flows, this theorem state that the circlation of the fluid around the curve C can be obtained by integrating the curl over the region bounded by C. If we think of C as the boundary of a small disc around a point, this explains the definition of curl: its value is approximately the rate at which the fluid "curls" around the point.

Equally interesting is the rate at which fluid passes through the boundary, given by $\int_C \mathbf{F}\cdot\mathbf{N}ds$. Using the expression $\mathbf{N}ds = dy\mathbf{I} - dx\mathbf{J}$, and $\mathbf{F} = P\mathbf{I} + Q\mathbf{J}$, we have

Proposition 18.8 *(Gauss' Divergence Theorem in the Plane). Let D be a region with boundary ∂D. Let* **F** *be a vector field defined on D. Then*

$$\oint_{\partial D} \mathbf{F}\cdot\mathbf{N}ds = \oint_{\partial D}(-Qdx + Pdy) = \int\int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \int\int_D \text{div } \mathbf{F}dA\ . \tag{18.80}$$

This is interpreted as saying (in terms of fluid flow) the rate of change of the amount of fluid inside the region D is equal to the flux of the fluid through the boundary.

Example 18.15 Let D be the disc of radius 1 centered at the point (0,1), and let C be its boundary oriented counter clockwise. Suppose $\mathbf{V} = -y\mathbf{I}$ is the velocity field of a flow in the upper half plane. Calculate the circulation along C and the flux through C.

First of all, we see that the the fluid is moving from right to left along the lines $y =$ const at speed proportional to the distance to the x-axis. Since fluid enters the disc from the right along any such line at the same speed as it leaves the disc, we should expect the flux to be zero. On the other hand, the fluid is moving to the left faster on the upper part of the circle (which is oriented to the left) than on the lower part of the circle, so we should expect a positive circulation. According to Stokes' theorem, the circulation is

$$\oint_C \mathbf{V}\cdot\mathbf{T}ds = \int\int_D \text{curl } \mathbf{V}\cdot\mathbf{K}dA\ . \tag{18.81}$$

Now, since curl $\mathbf{V} = \mathbf{K}$, this becomes simply

$$\oint_C \mathbf{V}\cdot\mathbf{T}ds = \int\int_D dA = \pi\ , \tag{18.82}$$

the area of D. According to the Divergence theorem, the flux out of D is

$$\oint_C \mathbf{F}\cdot\mathbf{N}ds = \int\int_D \text{div } \mathbf{F}dA = 0\ , \tag{18.83}$$

since the divergence of **V** is zero.

As a verification of these theorems, we also compute the line integrals. For that we use this parametrization of C: $\mathbf{X}(t) = \cos t\mathbf{I} + (1+\sin t)\mathbf{J}$. Then $d\mathbf{X} = (-\sin t\mathbf{I} + \cos t\mathbf{J})dt$, and since $\mathbf{V} = -y\mathbf{I} = -(1+\sin t)\mathbf{I}$ along C, we have

$$\oint_C \mathbf{V}\cdot d\mathbf{X} = \int_0^{2\pi}(1+\sin t)(\sin t)dt = \int_0^{2\pi}\frac{1}{2}dt = \pi \tag{18.84}$$

Now, to calculate the flux through C out of D, we have $\mathbf{N}ds = \cos t\mathbf{I} + \sin t\mathbf{J}$, and

$$\oint_C \mathbf{V}\cdot\mathbf{N}ds = \int_0^{2\pi} -(1+\sin t)(\cos t)dt = 0\ . \tag{18.85}$$

A simple application of Green's theorem leads to a way of calculating area by line integrals.

Proposition 18.9 *Let D be a region in the plane. Then the area of D is given by any of these line integrals over its boundary, ∂D:*

$$Area(D) \ = \oint_{\partial D} xdy = -\oint_{\partial D} ydx = \frac{1}{2}\oint_{\partial D} -ydx + xdy\ , \tag{18.86}$$

for in each of these cases the form $\partial Q/\partial y - \partial P/\partial x = 1$.

Example 18.16 Find the area of the region R bounded by the curves $y = x^2$ and $y = 1$.

We do this using Green's theorem. The boundary of R is in two pieces: $C_1 : y = 1$, with x going from 1 to -1, and $C_2 : y = x^2, -1 \le x \le 1$. Since $dy = 0$ on C_1, we have

$$Area = \oint_{\partial R} xdy = \int_{C_2} xdy = \int_{-1}^{1} x(2xdx) = \frac{4}{3}\ . \tag{18.87}$$

Example 18.19. We can verify that the area of an ellipse with major radius a and minor radius b is πab by Green's theorem and this parametrization of the boundar of the ellipse:

$$x = a\cos t\ , \quad y = b\sin t\ , \quad 0 \le t \le 2\pi\ . \tag{18.88}$$

Then

$$Area = \frac{1}{2}\oint_{\partial E} -ydx + xdy = \frac{1}{2}\int_0^{2\pi}(-b\sin t)(-a\sin t)dt + (a\cos t)(b\cos t)dt = \tag{18.89}$$

$$\frac{1}{2}\int_0^{2\pi} ab(\sin^2 t + \cos^2 t)dt = \pi ab\ . \tag{18.90}$$

§18.5. Stokes' and Gauss' theorems in three dimensions

When we move from two to three variables, the two interpretations of Green's theorem become two quite different theorems. Stokes' theorem relates integration on a surface with an integral on its bounding curve, and Gauss' theorem relates integration over a region with an integral on its bounding surface. We shall state these theorems and illustrate their use through examples, but shall not attempt to give proofs.

§18.5.1 Surface Integrals

Let $\mathbf{F}$ be the velocity field of a flow in three dimensions, and S a surface in the region of flow. We want to calculate the rate at which fluid is passing through the surface - this is called the **flux** of the flow through S. Take a small rectangle of area ΔS on the surface. In an in interval of time of length Δt, the fluid which passes through the sruface is very nearly that inside the parallelipiped whose base is the rectangle and whose side is the vector $\mathbf{V}\Delta t$. This volume is $\Delta V = (\mathbf{F}\cdot\mathbf{N})\Delta S \Delta t$, so

$$\frac{\Delta V}{\Delta t} = (\mathbf{F}\cdot\mathbf{N})\Delta S\ . \tag{18.91}$$

Now, if we sum these terms over a grid of rectangles on S, and take the limit as the grid becomes fine we get

Proposition 18.10 *Let* $\mathbf{F}$ *be a vector field defined in a neighborhood of the surface S. Choose a normal* $\mathbf{N}$ *to S. The* **flux** *of* $\mathbf{F}$ *over S in the direction* $\mathbf{N}$ *is*

$$Flux = \int\int_S (\mathbf{F}\cdot\mathbf{N})dS\ . \tag{18.92}$$

In order to calculate this, we assume that the surface S is given parametrically by $\mathbf{X} = \mathbf{X}(u,v)$, for (u,v) in a region R in u,v space. We have

$$\mathbf{N} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{|\mathbf{X}_u \times \mathbf{X}_v|}\ , \quad dS = |\mathbf{X}_u \times \mathbf{X}_v| du dv\ , \tag{18.93}$$

so

$$Flux = \int\int_S (\mathbf{F}\cdot\mathbf{N})dS = \int\int_R \mathbf{F}\cdot(\mathbf{X}_u \times \mathbf{X}_v) du dv\ . \tag{18.94}$$

Example 18.17 Let $\mathbf{F} = z^2\mathbf{I} + \mathbf{J} + x^2\mathbf{K}$, and H the upper hemisphere $x^2+y^2+z^2 = 1,\ z \geq 0$. Find the flux of $\mathbf{F}$ through H from the inside of the sphere.

We parametrize H using spherical coordinates:

$$H: \quad \mathbf{X}(\phi,\theta) = \cos\theta\sin\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\phi\mathbf{K} \tag{18.95}$$

for $0 \leq \phi \leq \pi/2,\ 0 \leq \theta \leq 2\pi$. Differentiating:

$$\mathbf{X}_\phi = \cos\theta\cos\phi\mathbf{I} + \sin\theta\cos\phi\mathbf{J} - \sin\phi\mathbf{K}\ , \tag{18.96}$$

$$\mathbf{X}_\theta = -\sin\theta\sin\phi\mathbf{I} + \cos\theta\sin\phi\mathbf{J}\ . \tag{18.97}$$

Check that the direction through H from the interior of the sphere is that of $\mathbf{X}_\phi \times \mathbf{X}_\theta$. Thus we must compute

$$\mathbf{F}\cdot(\mathbf{X}_\phi \times \mathbf{X}_\theta) = \det\begin{pmatrix} \cos^2\phi & 1 & \cos^2\theta\sin^2\phi \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \end{pmatrix} \tag{18.98}$$

(18.99) $$= \cos^2\phi\sin^2\phi\cos\theta - \sin^2\sin\theta + \sin^3\phi\cos\phi\cos^2\theta \ .$$

To calculate the integral (18.94), we first integrate with respect to θ. The first two terms integrate to zero, and since $\int_0^{2\pi}\cos^2\theta d\theta = \pi$, we obtain

(18.100) $$\int\int_S (\mathbf{F}\cdot\mathbf{N})dS = \pi\int_0^{\pi/2}\sin^3\phi\cos\phi d\phi = \frac{\pi}{4} \ .$$

§18.5.2 Stokes' theorem

Now, suppose that $\mathbf{F}$ is a vector field defined on a surface S in three dimensions, and S is bounded by a curve, denoted ∂S. As in two dimensions, Stokes' theorem relates the circulation about ∂S with the integral of curl $\mathbf{F}$ on S. For this to work we have to be sure that the direction of integration on ∂S is consistent with the choice of normal to S.

Proposition 18.11 *(Stokes' Theorem). Suppose that* $\mathbf{F}$ *is a vector field defined on the surface* S *with the boundary* ∂S*. Choose the direction of the tangent* $\mathbf{T}$ *to* ∂S *and the normal* $\mathbf{N}$ *to the surface so that the vector* $\mathbf{N}\times\mathbf{T}$ *points into the surface* S*. Then*

(18.101) $$\int_{\partial S}\mathbf{F}\cdot d\mathbf{X} = \int\int_S \text{curl}\ \mathbf{F}\cdot\mathbf{N}dS \ .$$

Example 18.18 Let S be the part of the plane $z = 2x+3y+z = 12$ which lies in the first quadrant. Let $\mathbf{F} = y\mathbf{I}+z\mathbf{J}+x\mathbf{K}$. Verify Stokes' theorem.
We want to calculate both sides of (18.101) and see that they agree. First, the surface integral. We write the surface parametrically as

(18.102) $$\mathbf{X}(x,y) = x\mathbf{I}+y\mathbf{J}+(12-2x-3y)\mathbf{K} \ ,$$

for (x,y) in the triangle T with vertices (0,0),(6,0), (0,4). We'll need the partial derivatives

(18.103) $$\mathbf{X}_x = \mathbf{I}-2\mathbf{K} \ , \quad \mathbf{X}_y = \mathbf{J}-3\mathbf{K} \ .$$

Now, we calculate curl $\mathbf{F} = -\mathbf{I}-\mathbf{J}-\mathbf{K}$, so

(18.104) $$\mathbf{F}\cdot(\mathbf{X}_u\times\mathbf{X}_v) = \det\begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{pmatrix} = -6 \ .$$

Then, using (18.94)

(18.105) $$\int\int_S \text{curl}\ \mathbf{F}\cdot\mathbf{N}dS = -6\int\int_T dxdy = -72 \ ,$$

since the area of T is 12.

Now, to calculate the boundary integral, we represent the boundary as composed of the three line segments

$$C_1 : 0 \le x \le 6, \quad z = 12 - 2x \quad y = 0\ ,; \quad dz = -2dx\ , dy = 0\ , \tag{18.106}$$

$$C_2 : 0 \le y \le 4, \quad x = \frac{12-3y}{2}\ , \quad z = 0; \quad dx = -\frac{3}{2}dy,\ dz = 0\ , \tag{18.107}$$

$$C_3 : 0 \le z \le 12, \quad y = \frac{12-z}{3}\ , \quad x = 0; \quad dy = -\frac{dz}{3},\ dx = 0\ . \tag{18.108}$$

Then, recalling that $\mathbf{F} = y\mathbf{I} + z\mathbf{J} + x\mathbf{K}$:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{X} = \int_0^6 x(-2dx) = -36\ , \tag{18.109}$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{X} = \int_0^4 y(-\frac{3}{2}dy) = -12\ , \tag{18.110}$$

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{X} = \int_0^{12} z(-\frac{dz}{3}) = -24\ . \tag{18.111}$$

The sum of these is -72, so Stokes' theorem is verified.

Example 18.19 Calculate $\int_C -ydx + xdy + dz$ where C is the curve of intersection of the hyperboloid $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$.

Let $\mathbf{F} = -y\mathbf{I} + x\mathbf{J} + \mathbf{K}$. Then this can be viewed as the integral $\mathbf{F} \cdot d\mathbf{X}$ over the boundary of the piece H of the hyperboloid lying over the disc of radius 1 in the x,y-plane. We calculate that curl $\mathbf{F} = 2\mathbf{K}$, so the integral is, by Stokes' Theorem

$$\int\int_H 2\mathbf{K} \cdot \mathbf{N} dS\ . \tag{18.112}$$

Now, we can parametrize H by $X(x,y) = x\mathbf{I} + y\mathbf{J} + (x^2 - y^2)\mathbf{K}$, with $\mathbf{X}_x = \mathbf{I} + 2x\mathbf{K}$, $\mathbf{X}_y = \mathbf{J} - 2y\mathbf{K}$, so that

$$\int\int_H 2\mathbf{K} \cdot \mathbf{N} dS = \int\int_{x^2+y^2 \le 1} 2\mathbf{K} \cdot (\mathbf{I} + 2x\mathbf{K}) \times \mathbf{J} - 2y\mathbf{K}) dxdy = \int\int_{x^2+y^2} 2dxdy = 2\pi\ , \tag{18.113}$$

since the area of the disc of radius 1 is π.
If we parametrize the curve by $\mathbf{X}(t) = \cos t\mathbf{I} + \sin t\mathbf{J} + (\cos^2 t - \sin^2 t)\mathbf{K}$, $0 \le t \le 2\pi$ and calculate directly, we again get 2π.

§18.5.3 Gauss' theorem

Now, suppose that R is a region in three dimensions, and the boundary of R is a surface which we shall denote as ∂R. If we have a fluid in flow, just as in 2 dimensions we expect Gauss' theorem to hold: the calculation of the rate of expansion of the fluid in R, which is the integral of the divergence, is the same as the flux through ∂R.

Proposition 18.12 *Gauss' theorem. Let* $\mathbf{F}$ *be a vector field defined on the region* R*. We denote the boundary of* R *as* ∂R*, and take the normal to be the exterior normal* $\mathbf{N}$*. Then*

$$\int\int_{\partial R} \mathbf{F}\cdot\mathbf{N}dS = \int\int\int_R \text{div}\,\mathbf{F}dV \ . \tag{18.114}$$

Example 18.20 Let R be the region inside the cone $z^2 = x^2+y^2$, bounded by the planes $z=0$ and $z=2$. Let $\mathbf{F} = x\mathbf{I}+y\mathbf{J}+z\mathbf{K}$. Verify the divergence theorem in this context.

We easily calculate div $\mathbf{F}=3$, so the right hand side of (18.94) is 3 times the volume of the cone, so

$$\int\int\int_R \text{div}\,\mathbf{F}dV = 3\left(Volume\left(R\right)\right) = 3\frac{\pi r^2 h}{3} = 8\pi \ , \tag{18.115}$$

since $r=2,\ h=2$.

To calculate the boundary integral, we turn to cylindrical coordinates, because of the symmetry around the z-axis. The boundary has two pieces: the disc $D:\ z=1, r\le 1$, and the surface of the cone $S:\ z=r\le 1$. We can see that the integral over S is zero, since the vector field $\mathbf{F}$ is tangent to the cone (it is the tangent vector to the line $z=r,\theta=\theta_0$ which lies on the cone). Thus we need only calculate the boundary integral over D. Since D lies on the plane $z=2$, its normal is $\mathbf{K}$. Thus since $\mathbf{F}\cdot\mathbf{K}=z=2$ on the plane $z=2$,

$$\int\int_{\partial R} \mathbf{F}\cdot\mathbf{N}dS = \int_0^{2\pi}\int_0^2 2rdrd\theta = 4\pi\frac{r^2}{2}\Big|_0^2 = 8\pi \ . \tag{18.116}$$

One of the main points of the divergence theorem is that informed use of the geometry involved simplifies what could otherwise be a complicated calculation. For example, if we did not observe that $\mathbf{F}$ is orthogonal to the normal to the cone, we'd have to do the calculation. Just to illustrate the methods we do it. First of all, we parametrize the cone using cylindrical coordinates:

$$S:\quad \mathbf{X} = r\cos\theta\mathbf{I}+r\sin\theta\mathbf{J}+r\mathbf{K}\ , 0\le\theta\le 2\pi,\ r\le 2 \tag{18.117}$$

and, differentiating, we find

$$\mathbf{X}_r = \cos\theta\mathbf{I}+\sin\theta\mathbf{J}+\mathbf{K}\ ,\quad \mathbf{X}_\theta = -r\sin\theta\mathbf{I}+r\cos\theta\mathbf{J}\ . \tag{18.118}$$

On the surface, in these coordinates $\mathbf{F} = r\cos\theta\mathbf{I}+r\sin\theta\mathbf{J}+r\mathbf{K}$. Now we calculate $\det(\mathbf{F},\mathbf{X}_r,\mathbf{X}_\theta)=0$, or we observe that since $\mathbf{F}=r\mathbf{X}_r$, the determinant must be zero.

Example 18.21 Return to example 20, and note that the divergence of that vector field is 0. By applying the divergence theorem, where R is the region bounded by H and the x,y-plane we can replace the integration of example 20 by the easier integration over the planar part of the boundary of H. That surface is the disc $D: \ x^2+y^2 \leq 1, z=0$. The normal (pointing outside of the region R) is $-\mathbf{K}$ and on this disc, $\mathbf{F} = \mathbf{J} + x^2\mathbf{K}$. Thus

$$\int\int_D \mathbf{F}\cdot\mathbf{N}dS = -\int\int_D x^2 dA = -\int_0^{2\pi}\int_0^1 r^2\cos^2\theta r dr d\theta = -\frac{\pi}{4} \ . \tag{18.119}$$

Now, for this example, the divergence theorem tells us that

$$\int\int_H \mathbf{F}\cdot\mathbf{N}dS + \int\int_D \mathbf{F}\cdot\mathbf{N}dS = 0 \ , \tag{18.120}$$

which gives the result $\int\int_H \mathbf{F}\cdot\mathbf{N}dS = \pi/4$.