CHAPTER 11

Conics and Polar Coordinates

§11.1. Quadratic Relations

We will see that a curve defined by a quadratic relation between the variables x, y is one of these three curves: a) parabola, b) ellipse, c) hyperbola. There are other possibilities, considered degenerate. For example the graph of the equation $x^2 + y^2 = a$ we know to be a circle, if a > 0. But if a = 0, the graph is just the point (0,0), and if a < 0, there is no graph. Similarly the equation $x^2 - y^2 = a$ describes a hyperbola if $a \neq 0$, but if a = 0, we get the two lines $x = \pm y$.

First we list the **standard forms** of the basic curves. These are standard in the sense that any other curve given by a quadratic equation is obtained from one of these by moving the curve in the plane by translating and/or rotating.

The Parabola. The standard form is one of these:

$$y = ax^2, \qquad x = ay^2$$

The sign of *a* determines the orientation of the parabola. We have these four possibilities:

 $y = ax^2, a > 0$ $y = ax^2, a < 0$ $x = ay^2, a > 0$ $x = ay^2, a < 0$ Figure 11.1 Figure 11.2 Figure 11.3 Figure 11.4





The magnitude of *a* determines the spread of the parabola: for |a| very small, the curve is narrow, and as |a| gets large, the parabola broadens. The origin is the **vertex** of the parabola. In the first two cases, the *y*-axis is the **axis** of the parabola, in the second two cases it is the *x*-axis. The parabola is symmetric about its axis.

The Ellipse The standard form is

(11.2)
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The values x can take lie between -a and a and the values y can take lie between -b and b. If a > b (as shown in figure 11.5), the **major axis** of the ellipse is the x-axis, the **minor axis** is the y-axis and the points $(\pm a, 0)$ are its **vertices**.



If a < b (as shown in figure 11.6), the major axis of the ellipse is the *y*-axis, x = 0 is the minor axis, and the points $(0, \pm b)$ are its **vertices**.

Of course, if a = b, the curve is the circle of radius a, and there are no special vertices or axes.

The Hyperbola. The standard form is one of these:

(11.3)
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 ,$$

corresponding to figures 11.7 and 11.8.



The *x*-axis is the **axis** of the first hyperbola. The points $(\pm a, 0)$ are the vertices of the hyperbola; for *x* between these values, there corresponds no point on the curve. We similarly define the axis and vertices of the hyperbola of figure 11.8.

The lines

(11.4)
$$y = \pm \frac{b}{a}x$$

are the **asymptotes** of the hyperbola, in the sense that, as $x \to \infty$, the curve gets closer and closer to these lines. We see this by dividing the defining equation by x^2 , and consider what happens as $x \to \infty$. For example, using the first equation, we get



which we can rewrite as

(11.6)
$$\frac{a^2}{b^2}\frac{y^2}{x^2} = 1 - \frac{1}{x^2} ,$$

so that, as |x| gets large, the hyperbola approaches the graph of

(11.7)
$$\frac{a^2}{b^2}\frac{y^2}{x^2} = 1$$

which amounts to the two equations $y = \pm (b/a)x$. Figure 11.9 shows these asymptotes.

Now, the general quadratic relation between x and y is

(11.8)
$$Ax^{2} + By^{2} + Cxy + Dx + Ey + F = 0$$

If C = 0, then by completing the square in both x and y we are led to an equation which looks much like one of the standard forms, but with the center removed to a new point (x_0, y_0) . If $C \neq 0$, the situation is more difficult: a rotation of the figure is also required to get it into standard form. We will discuss this no further, and consider only the case C = 0. First, some examples:







Proposition 11.1 *The equation*

(11.15)
$$Ax^2 + By^2 + Dx + Ey + F = 0$$

can be put into one of the following forms by completing the square:

a) $(parabola): y - y_0 = A(x - x_0)^2$, if B = 0. The vertex of the parabola is at (x_0, y_0) , and the axis is the line $x = x_0$.

b) (parabola): $x - x_0 = C(y - y_0)^2$ if A = 0. The vertex of the parabola is at (x_0, y_0) , and the axis is the line $y = y_0$.

c) (ellipse) $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$ if A and B are of the same sign The center of the ellipse is at (x_0, y_0) , and its axes are the lines $x = x_0$, $y = y_0$. d) (hyperbola): $\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1$ or $\frac{(y-y_0)^2}{b^2} - \frac{(x-x_0)^2}{a^2} = 1$ if A and B are of different signs. The center of the hyperbola is (x_0, y_0) , and its axes are the lines $x = x_0$, $y = y_0$. e) If both A and B are zero, the curve is a line. The following degenerate cases may also result:

(11.16) $A(x-x_0)^2 + B(y-y_0)^2 \le 0 : \text{ no graph or just the point } (x_0, y_0).$

(11.17) $A(x-x_0)^2 - B(y-y_0)^2 = 0 : \text{ two lines crossing at } (x_0, y_0).$

Example 11.4 Finally, just to illustrate the situation of a quadratic whose coefficient of xy is nonzero, we consider the curve xy - 1 = 0. This curve is symmetric about the lines $y = \pm x$, and has the asymptotes x = 0, y = 0. This appears to be a hyperbola with major axis the line x = y. In fact, if we make the linear change of variables x = u + v, y = u - v, this becomes the curve $u^2 - v^2 = 1$ in the new variables. (This change of variables represents a rotation by 45° , with a slight change of scale.)

§11.2. Eccentricity and Foci

These curves are called the **conic sections** because they can be visualized as the intersection of a cone with a plane. We shall now consider another definition, dating from the ancient Greeks, which leads to important properties of the conics.

Fix a point F and a line L in the plane such that L does not go through F. Pick a positive number e. We consider the locus C of all points X in the plane such that

$$(11.18) |XF| = e|XL|$$

where |XY| means the distance from X to Y. e is the eccentricity of C; F the focus and L the directrix. Note that the curve C is symmetric about the line through the focus and perpendicular to the directrix. This is the **axis** of the curve. There is one point between F and L on C which is on this axis; this point is the **vertex** of C.

We now show that if e = 1, *C* is a parabola, if e < 1, *C* is an ellipse and if e > 1, *C* is a hyperbola. Let's take the axis of *C* to be the *x* axis, and place the vertex at the origin, *O*. Then the focus is some point (p,0); we take p > 0. Since |OF| = p, from 11.18 we find that the directrix is the line x = -p/e (see figure 11.14).



Now, for a point X = (x, y) on the curve, we have

(11.19) |XL| = x + p/e and $|XF| = \sqrt{(x-p)^2 + y^2}$

and so equation 11.18 in coordinates is given by

(11.20)
$$\sqrt{(x-p)^2 + y^2} = e(x+p/e) = ex+p \, .$$

Case e = 1. Squaring both sides we get

(11.21)
$$x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2$$
 simplifying to $y^2 = 4px$.

This of course is the standard form of a parabola. It also locates the focus and the directrix of a parabola.

Proposition 11.2 The focus of the parabola $y^2 = ax$ is a/4 units on one side of the vertex of the parabola along the axis, and and the directrix intersects the axis a/4 units on the other side.

Example 11.5 Find the vertex, focus and directrix of the parabola given by the equation $2x^2 + 6x - y + 4 = 0$.

First we put the equation in standard form. Completing the square, we have

(11.22)
$$2\left(x^2+3x+\frac{9}{4}\right)-\frac{9}{2}=y-4$$
, or $\left(x+\frac{3}{2}\right)^2=\frac{1}{2}\left(y+\frac{1}{2}\right)$.

Thus the vertex is at (-3/2, 1/2), the axis of the parabola is the line x = -3/2 and we have 4p = 1/2, so p = 1/8. Thus the focus is at (-3/2, (1/2) + (1/8)) = (-3/2, 5/8) and the directrix is the line y = 3/8.

Example 11.6 Find the equation of the parabola whose vertex is at (4,2) and whose directrix is the line x = -1. Find the focus of this parabola.

Since the directrix is a vertical line, the axis is horizontal, so the equation has the form

(11.23)
$$(y-2)^2 = 4p(x-4)$$
,

since the vertex is at (4,2). Now p is the distance between the vertex and the directrix, so p = 2 - (-1) = 5. Thus the equation of the parabola is

(11.24)
$$(y-2)^2 = 20(x-4)$$

The focus is 5 units to the right of the vertex, so is at (9,2).

Example 11.7 Find the equation of the parabola whose focus is the origin and whose vertex is at the point (a, 0) with a > 0.

The parabola has its axis the *x*-axis, and since the vertex is to the right of the focus, the parabola opens to the left. Thus the equation has the form

(11.25)
$$y^2 = -4p(x-a) \, ,$$

where p is the distance between focus and vertex. But that is a, so the equation is

(11.26)
$$y^2 = -4a(x-a)$$
.

Case $e \neq 1$. Squaring both sides of 11.20 gives us

(11.27)
$$x^2 - 2px + p^2 + y^2 = e^2(x^2 + 2px + p^2)$$

which simplifies to

(11.28)
$$(1-e^2)x^2 + y^2 - 2p(1+e)x = 0$$

Thus, if e < 1, this is an ellipse, and if e > 1 this is a hyperbola. Notice, because of symmetry in the minor axis, ellipses and hyperbolas have two foci; one on each side of the minor axis.

We now want show how to locate the foci of an ellipse given in standard form. Thus we start by putting 11.28 in standard form, and then compare it to the formula of proposition 11.1c. Dividing equation 11.28 by the coefficient of x^2 gives us

(11.29)
$$x^2 - \frac{2p}{1-e}x + \frac{y^2}{1-e^2} = 0$$

Now completing the square, we come to

(11.30)
$$\left(x - \frac{p}{1 - e}\right)^2 + \frac{y^2}{1 - e^2} = \frac{p^2}{(1 - e)^2}$$

Comparing this to

(11.31)
$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

we see that the center of the ellipse is at (p/(1-e), 0) and a = p/(1-e), $b^2 = (1-e^2)a^2$. Let *c* be the distance of the center from the focus. Then

(11.32)
$$c = \frac{p}{1-e} - p = e\frac{p}{1-e} = ea$$

and $c^2 = e^2 a^2 = a^2 - b^2$. Summarizing

Proposition 11.3 If an ellipse is in standard form (11.31), with a > b, then the foci of the ellipse are on the major axis, c units away from the center where

(11.33)
$$c^2 = a^2 - b^2$$

The eccentricity of the ellipse is given by the equations

(11.34)
$$b^2 = (1 - e^2)a^2$$
 or $c = ea$

The same arguments for the case e > 1, the hyperbola, lead to

Proposition 11.4 If a hyperbola is in standard form

(11.35)
$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1$$

then the foci of the hyperbola are on the major axis, c units away from the center where

(11.36)
$$c^2 = a^2 + b^2$$

The eccentricity of the hyperbola is given by the equations

(11.37)
$$b^2 = (e^2 - 1)a^2$$
 or $c = ea$

Example 11.8 Find the foci of the conic given by the equation $x^2 + 4y^2 - 2x = 8$. First, we complete the square to get the equation in standard form:

(11.38)
$$\frac{(x-1)^2}{3^2} + \frac{y^2}{(3/2)^2} = 1$$

This conic is an ellipse centered at (1,0), with major axis the x-axis, and $a^2 = 9$, $b^2 = 9/4$. Thus $c^2 = a^2 - b^2 = 9(3/4)$, so $c = (3/2)\sqrt{3}$. This is the distance of the foci from the center (along the major axis), so the foci are at $(1 \pm (3/2)\sqrt{3}, 0)$.

Example 11.9 Find the foci of the conic given by the equation $y^2 - x^2 + 4x = 13$. Complete the squares, and get the standard form

(11.39)
$$\frac{y^2}{3^2} - \frac{(x-2)^2}{3^2} = 1$$

This is a hyperbola with center at (2,0), and major axis the line x = 2. We have $c^2 = a^2 + b^2 = 18$, so $c = 3\sqrt{2}$ is the distance of the foci from the center along the line x = 2. Thus the foci are at $(2, \pm 3\sqrt{2})$. The vertices are at $(2, \pm 3)$.

Example 11.10 Find the equation of the ellipse centered at the origin, with a focus at (2,0) and a vertex at (3,0).

The equation of an ellipse centered at the origin is

(11.40)
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We are given a = 3, c = 2. Thus $b^2 = a^2 - c^2 = 5$, and the equation is

(11.41)
$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

§11.3. String and Optical Properties of the Conics

We have seen that the parabola can be defined as the locus of points X equidistant from a given point F and a given line L. The ellipse and the hyperbola have similar definitions.

Proposition 11.5 Given two points F_1 and F_2 and a number a greater than the distance between F_1 and F_2 , the locus of points X such that

$$(11.42) |XF_1| + |XF_2| = 2a$$

is an ellipse with foci at F_1 and F_2 and major axis of length 2a.

Given an ellipse in standard form, we can verify 11.42 by a lengthy algebraic computation. To show that 11.42 leads to the equation of an ellipse is another algebraic computation beginning this way. Choose coordinates so that the points F_1 and F_2 lie on the *x*-axis, and the origin is midway between the points. Then F_1 has coordinates (-c, 0), and F_2 has coordinates (c, 0) for some c < a. Let *X* have the coordinates (x, y). Then 11.42 becomes

(11.43)
$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

Eliminate the radicals to verify that we end up with a quadratic equation which is that of an ellipse. We have a similar description of the hyperbola:

Proposition 11.6 Given two points F_1 and F_2 and a positive number a, the locus of points X such that

$$(11.44) |XF_1| - |XF_2| = 2a$$

is a hyperbola with foci at F_1 and F_2 .

Actually, this is just the branch of the hyperbola which wraps around the focus F_2 ; the other branch is given by the equation

$$(11.45) |XF_2| - |XF_1| = 2a$$

The optical properties of the conics follow from these string characterizations. Let's start with the parabola. Suppose that the parabola is coated with a light-reflecting material. The rays of a beam of light originating far away along the axis of the parabola will approach the parabola along lines parallel to



its axis. According to the physics of the situation, the angle of reflection off the parabola is equal to the angle of incidence. The optical property of the parabola is that these reflected rays all meet at the focus.

Proposition 11.7 Let X be a point on the parabola, and T the tangent line to the parabola at X. Let L_F be the line from the focus to X, and L the line through X parallel to the axis of the parabola. Then the angle between T and L_F is equal to the angle between T and L.

What we want to show, referring to figure 11.15, is that $\gamma = \alpha$. From the figure we see that this amounts to showing that $\beta - \alpha = \alpha$. Let us think of the parabola as being traced out by a particle moving to the right at constant velocity 1. This expresses the coordinates (x, y) of the point *X* as functions of arc length *s*. The string property of the parabola tells us that

(11.46)
$$\sqrt{(x-c)^2 + y^2} = x + c$$

Differentiating with respect to arc length gives us

(11.47)
$$\frac{2(x-c)\frac{dx}{ds} + 2y\frac{dy}{ds}}{2\sqrt{(x-c)^2 + y^2}} = \frac{dx}{ds}$$

which simplifies to

(11.48)
$$\frac{x-c}{\sqrt{(x-c)^2+y^2}}\frac{dx}{ds} + \frac{y}{\sqrt{(x-c)^2+y^2}}\frac{dy}{ds} = \frac{dx}{ds}$$

Now we do a little trigonometry:

(11.49)
$$\cos\beta = \frac{x-c}{\sqrt{(x-c)^2 + y^2}} \quad \sin\beta = \frac{y}{\sqrt{(x-c)^2 + y^2}} \quad \cos\alpha = \frac{dx}{ds} \quad \sin\alpha = \frac{dy}{ds}$$



so (10) becomes

(11.50)
$$\cos\beta\cos\alpha + \sin\beta\sin\alpha = \cos\alpha$$

or $\cos(\beta - \alpha) = \cos \alpha$, from which we conclude $\beta - \alpha = \alpha$ as desired.

The optical property of the ellipse is that a ray of light emanating from one focus reflects off the ellipse so as to pass through the other focus.

Proposition 11.8 Let X be a point on the ellipse, and T the tangent line to the ellipse at X. Let L_1 be the line from the focus F_1 to X, and L_2 the line from the other focus F_2 to X. Then the angle between T and L_1 is equal to the angle between T and L_2 .

What we want to show, referring to figure 11.16, is that $\beta_2 + \alpha = \beta_1 - \alpha$. We start with the string property, written in the coordinates as shown in the figure:

(11.51)
$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

We now differentiate with respect to arc length, and arrive at

(11.52)
$$\frac{x+c}{\sqrt{(x+c)^2+y^2}}\frac{dx}{ds} + \frac{y}{\sqrt{(x+c)^2+y^2}}\frac{dy}{ds} + \frac{x-c}{\sqrt{(x-c)^2+y^2}}\frac{dx}{ds} + \frac{y}{\sqrt{(x-c)^2+y^2}}\frac{dy}{ds} = 0$$

We now make the substitutions with the trigonometric functions, but here we have to be careful: in our picture dy and x - c are negative, so since the sine and cosine are ratios of *lengths*, we have

(11.53)
$$\cos \beta_1 = -\frac{x-c}{\sqrt{(x-c)^2 + y^2}} \qquad \sin \alpha = \left|\frac{dy}{ds}\right| = -\frac{dy}{ds}.$$

Thus our equation becomes

(11.54)
$$\cos\beta_2\cos\alpha + \sin\beta_2(-\cos\alpha) + (-\cos\beta_1)\cos\alpha + \sin\beta_1(-\sin\alpha) = 0$$

or

(11.55)
$$(\cos\beta_2\cos\alpha - \sin\beta_2\cos\alpha) - (\cos\beta_1\cos\alpha + \sin\beta_1\sin\alpha) = 0$$

which is $\cos(\beta_2 + \alpha) - \cos(\beta_1 - \alpha) = 0$, so $\beta_2 + \alpha = \beta_1 - \alpha$ as desired

The optical property of the hyperbola is that a ray of light emanating from one focus reflects off the opposite branch of the hyperbola so as to appear to have come from the other focus.

Proposition 11.9 Let X be a point on the hyperbola, and T the tangent line to the ellipse at X. Let L_1 be the line from the focus F_1 to X, and L_2 the line from the other focus F_2 to X. Then the exterior angles between T and L_1 and between T and L_2 are equal



§11.4. Polar Coordinates



Often a problem can be seen as that of understanding the motion of a particle in the plane relative to a fixed point. In such a situation it is desirable to be able to describe a position in terms of the length and the direction of the line between the two points. These are the **polar coor-dinates** of the point. We consider the fixed point as the origin of these coordinates, and take the positive *x*-axis as the "zero" direction. Then any other direction is described by the angle between it and the positive *x* axis, which we denote as θ . The distance of a point on this line from the origin is denoted *r*. These equations relate the cartesian coordinates (*x*, *y*) with the polar coordinates *r*, θ :

(11.56)
$$x = r\cos\theta$$
, $y = r\sin\theta$, $r = \sqrt{x^2 + y^2}$, $\theta = \arctan\frac{y}{r}$

See figure 11.18

Polar coordinates have two pecularities which we need to get used to. Every value of (r, θ) determines a point in the plane. However, if r = 0, the point is the origin, and θ doesn't make sense. Secondly, the values (r, θ) and $(r, \theta + 2\pi)$, and in fact, $(r, \theta + 2n\pi)$ for any *n* give the same point. This ambiguity is sometimes of value: for example, when discussing the motion of a particle, *n* tells us how many times the particle has wound around the origin in the counterclockwise sense. Finally, it is also of convenience to let *r* take negative values, meaning a distance of |r| in the opposite direction of the ray θ . Thus (r, θ) and $(-r, \theta + \pi)$ determine the same point. We now consider the graphs of equations in polar coordinates.

Example 11.11 The equation r = a, for a > 0 is satisfied by all points of distance *a* from the origin, so is polar equation of the circle of radius *a* centered at the origin.





Example 11.13 $r = a\theta$ describes the motion of a point which rotates around the origin at angular velocity 1 while moving out along the ray at velocity *a*. This is the **Archimedean spiral** and is shown in figure 11.19.

Example 11.14 $r = e^{a\theta}$ is another spiral, however, the point moves out along the ray at a rate exponential in the rate of rotation. This is the **logarithmic spiral** and is shown in figure 11.20.

Example 11.15 The equation $r = a \cos \theta$ is the circle of diameter *a* with center on the *x*-axis which goes through the origin. For, if we multiply by *r* we get $r^2 = ar \cos \theta$, which can now be written in cartesian coordinates (using (11.56)) as

(11.57)
$$x^2 + y^2 = ax$$
 or $\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$.

Given an equation of the form $r = r(\theta)$, we can often trace out the graph by just studying the behavior of the function $r(\theta)$. Let's redo example 11.15 this way. We have this table



It will be useful for you to follow the following discussion along the curve in figure 11.21. Between 0 and $\pi/2$ the point is in the first quadrant, and as the angle increases it moves toward the origin, reaching there at $\theta = \pi/2$. Then for θ between $\pi/2$ and π , the point is in the fourth quadrant (because r < 0), steadily moving away from the origin until we reach the point we've started with. This looks like a circle, and the argument above (in example 11.15) shows that it is. Note that as θ moves from π to 2π the circle is retraced.

Example 11.16 Similarly, the equation $r = a\cos(\theta - \theta_0)$ is the circle through the origin of radius *a* with center on the ray of angle θ_0 . This amounts to the assertion that any equation of the form

(11.58)
$$r = a\cos\theta + b\sin\theta$$

is a circle with the origin the endpoint of one of its diameters (see practice problem 12.1).

Example 11.17 If we are given the equation of a curve in cartesian coordinates, we can find its equation in polar coordinates through the substitution $x = r \cos \theta$, $y = r \sin \theta$. For example

(11.59) Equation of a line:
$$r = \frac{c}{a\cos\theta + b\sin\theta}$$
.

For, the general equation of a line is ax + by = c. After substitution this becomes

(11.60)
$$ar\cos\theta + br\cos\theta = c_{\rm s}$$

which becomes the above when we solve for r.

Example 11.18 The polar equation of a conic of eccentricity *e*, focus at the origin and directrix the line x = -d is

(11.61) Equation of a Conic:
$$r = \frac{ed}{1 - e\cos\theta}$$

To show 11.61, we start with the defining relation |XF| = e|XL|, referring to figure 11.22. In polar coordinates this gives us

(11.62)
$$r = e(d+x) = e(d+r\cos\theta)$$

Solving for r brings us to (11.61). If the figure is rotated by θ_0 , we just replace θ with $\theta - \theta_0$

Example 11.19 $r = a\cos 2\theta$. We first construct the table:

$$\theta \ 0 \ \frac{\pi}{4} \ \frac{\pi}{2} \ \frac{3\pi}{4} \ \pi$$

 $r \ a \ 0 \ -a \ 0 \ a$

Follow this discussion along the graph shown in figure 11.23. This time the curve starts (at $\theta = 0$) at r = a, and decreases to zero by $\theta = \pi/4$. Between $\pi/4$ and $\pi/2$, r is negative, so the curve is in the third quadrant, and as θ rotates counterclockwise, r moves away from the origin finally to r = -a for $\theta = \pi/2$. As θ increases from $\pi/2$ the point continues to move toward the origin (in the fourth quadrant), arriving there at $\theta = 3\pi/2$. Moving on, r becomes positive, so we enter the second quadrant with the distance from the origin steadily increasing until, at $\theta = \pi$ we are at r = a. Since $\cos \theta$ is an even function, as we move from π to 2π (or what is the same, from $-\pi$ to 0), we just get the same curve, reflected in the *x*-axis. This is the **four-petalled rose** shown in figure 11.24.



Example 11.20 $r = a \cos 3\theta$ is a three-petalled rose. Construct the table of important values between 0 and π and argue as in example 11.19. The table is



That completes the rose; as we proceed from π to 2π we traverse the rose again. See figure 11.25.



We conclude

Proposition 11.10 *The graph of the equation* $r = a\cos(n\theta)$ *or* $r = a\sin(n\theta)$ *is a 2n-petalled rose if n is even, and an n petalled rose if n is odd (traversed twice).*

Limaçons. These are the curves defined by the equation $r = a + b \cos \theta$. First, we consider the case:a > b. We have the table

$$\begin{array}{ccccccc} \theta & 0 & \frac{p_l}{2} & \pi & \frac{3\pi}{2} & 2\pi \\ r & a+b & a & a-b & a & a+b \end{array}$$

leading to the graph shown in figure 11.26.

As *b* gets closer and closer to *a*, the value of *r* for $\theta = \pi$ goes to zero. Thus when a = b, we get the graph shown in figure 11.27, called the **cardioid**.



Then as b goes beyond a, r becomes negative as θ gets near π , and there is an inner loop of the limaçon

Example 11.21 $r = 2 + 4\cos\theta$. Our table is this:

When $\cos \theta = -1/2$, that is, for $\theta = \pm 2\pi/3$, the value of r is zero, and between these two values r is negative. We get the graph shown in figure 11.28. We have drawn the curve so that it is tangent to $\theta = \pm \pi/3$ for those values of θ . This is correct, as we will show in the next section.

Finally, it is important to note that if the function $\cos \theta$ is replaced by $-\cos \theta$ the curve is reflected in the y-axis, and if it is replaced by $\pm \sin \theta$, it is rotated by a right angle.

§11.5. Calculus in polar coordinates

Arc length

Consider the curve given in polar coordinates by the equation $r = r(\theta)$. We can calculate the differential *ds* of arc length by the differential triangle in polar coordinates using this diagram.



The length of the arc of the circle of radius r subtended by the angle $d\theta$ is $rd\theta$. The differential triangle is thus a right triangle with side lengths dr and $rd\theta$. By the pythagorean theorem

$$ds^2 = dr^2 + r^2 d\theta^2$$

Example 11.22 Find the length of the curve $r = \theta^2$ from 0 to 2π .

This curve is a spiral whose distance from the origin increases as the square of the angle. We have $dr = 2\theta d\theta$, so

(11.64)
$$ds^{2} = dr^{2} + r^{2}d\theta^{2} = 4\theta^{2}d\theta^{2} + \theta^{4}d\theta^{2} = \theta^{2}(4+\theta^{2})d\theta^{2}$$

and thus the length is

(11.65)
$$\int_{0}^{2\pi} ds = \int_{0}^{2\pi} \theta \sqrt{4 + \theta^2} d\theta = \frac{1}{3} \left(4 + \theta^2\right)^{3/2} \Big|_{0}^{2\pi} = \frac{1}{3} \left(\left(4 + 4\pi^2\right)^{3/2} - 4^{3/2}\right)$$

Area

To caculate the area enclosed by a curve given, in polar coordinates, by $r = r(\theta)$, we calculate the differential of area, using figure 11.30.



The area of the wedge given by the increment $d\theta$ is $(1/2)r^2d\theta$. To see this, we start with the area of the circle of radius $r : A = \pi r^2$. Now an angle α subtends a segment of the circle which is the $(\alpha/2\pi)$ th part of the full circle, thus the area of that segment is $(1/2)r^2\alpha$. Thus, for $\alpha = d\theta$, we get

$$dA = \frac{1}{2}r^2d\theta.$$

Figure 11.30

Example 11.23 Find the area enclosed by the cardioid $r = 3(1 + \sin \theta)$.

The area is

(11.67)
$$Area = \frac{1}{2} \int_0^{2\pi} \left[3(1+\sin\theta)^2 \right] d\theta = \frac{9}{2} \int_0^{2\pi} (1+2\sin\theta+\sin^2\theta) d\theta$$

Now, we know that the integral of $\sin \theta$ over an entire period is zero, so we can neglect the middle term. We now use the double angle formula for the last term, and drop the integral of $\cos(2\theta)$ for the same reason:

(11.68)
$$Area = \frac{9}{2} \int_0^{2\pi} \left(1 + \frac{1 - \cos(2\theta)}{2} \right) d\theta = \frac{9}{2} \int_0^{2\pi} \frac{3}{2} d\theta = \frac{27}{2} \pi.$$

Example 11.24 Find the area inside one petal of the rose $r = \sin 3\theta$.

At $\theta = 0$ we have r = 0, but then as the angle rotates, r increases to its maximum at $3\theta = \pi/2$, and then decreases back to zero for $3\theta = \pi$. Thus one petal is spanned as θ ranges from 0 to $\pi/3$. We now calculate;

(11.69)
$$Area = \frac{1}{2} \int_0^{\pi/3} \sin^2(3\theta) \, d\theta = \frac{1}{2} \int_0^{\pi/3} \left(\frac{1 - \cos(6\theta)}{2}\right) d\theta = \frac{1}{2} \left(\frac{\theta}{2} - \frac{\cos(6\theta)}{12}\right) \Big|_0^{\pi/3} = \frac{\pi}{12} \, .$$



Tangents Given the polar equation $r = r(\theta)$ of a curve, we can find the tangent at any point as follows. First of all, the cartesian coordinates are given by $x = r(\theta) \cos \theta$, $y = r(\theta) \sin \theta$. If *m* is the slope of the tangent line, we have, by the chain rule

(11.70)
$$m = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\theta\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\frac{dr}{d\theta}}$$

Notice that, as $r \to 0$, the right hand side approaches $\tan \theta$. Thus, if θ_0 is a value for which r = 0, then the curve approaches the origin along the ray $\theta = \theta_0$.

Example 11.25 What is the slope of the tangent to the inner loop of the limacon

$$(11.71) r = 2 + 5\cos\theta$$

at the origin?

We find the values of θ for which r = 0:

(11.72)
$$2 + 5\cos\theta = 0 \quad \text{or} \quad \cos\theta = -\frac{2}{5}$$

so that $\theta = \pm 0.63\pi$ radians or 113.6°.