Calculus II
Practice Final Exam, Answers

1. Differentiate:

a) $f(x) = \ln(\sin(e^{2x}))$.

Answer. This is an exercise in the chain rule:

$$f'(x) = \frac{1}{\sin(e^{2x})}\cos(e^{2x}) \cdot 2e^{2x} = 2e^{2x}\cot(e^{2x})$$

b) $g(x) = x\tan^{-1}(x^2)$.

Answer. This is an exercise in the product rule:

$$g'(x) = \tan^{-1}(x^2) + x\frac{2x}{1+(x^2)^2} = \tan^{-1}(x^2) + \frac{2x^2}{1+x^4}$$

c) $h(x) = e^{\ln x}$.

Answer. This is an exercise in the definition of ln : $e^{\ln x} = x$, so $h'(x) = 1$.

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2. Find the integrals:

a) $\int u^2(u-1)^5 du$

Answer. Let $v = u-1,\ dv = du$. Then

$$\int u^2(u-1)^5 du = \int (v+1)^2 v^5 dv = \int (v^7 + 2v^6 + v^5) dv$$

$$= \frac{1}{8}(u-1)^8 + \frac{2}{7}(u-1)^7 + \frac{1}{6}(u-1)^6 + C\ .$$

b) $\int x(\ln x) dx$

Answer. Let $u = \ln x,\ dv = xdx$ so that $du = dx/x,\ v = x^2/2$, and we can integrate by parts:

$$\int x(\ln x) dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2} x dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$$

c) $\int \frac{e^x}{1+e^x} dx$

Answer. Let $u = 1+e^x,\ du = e^x dx$:

$$\int \frac{e^x}{1+e^x} dx = \int \frac{du}{u} = \ln(1+e^x) + C$$

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3. Integrate $\int \frac{3x+1}{x(x^2+1)} dx$

Answer. First we must find the partial fractions expansion of the integrand:

$$\frac{3x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+x(Bx+C)}{x(x^2+1)}$$

The numerators on the left and right are equal. Evaluating at $x=0$, we find $A=1$. Equating the coefficients of $x^2:\ 0=A+B$, so $B=-1$. Finally, equating the coefficients of $x:\ C=3$. Thus

$$\int \frac{3x+1}{x(x^2+1)}dx = \int \frac{dx}{x} - \int \frac{xdx}{x^2+1} + 3\int \frac{dx}{x^2+1} = \ln x - \frac{1}{2}\ln(x^2+1) + 3\arctan x + C$$

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4. Integrate $\int \frac{x^2+1}{(x-1)(x-2)(x-3)}dx$

Answer. First we must find the partial fractions expansion of the integrand:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$= \frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

Evaluating at $x=1$, we get $2=A(-1)(-2)$, at $x=2$, we get $5=B(1)(-1)$, at $x=3$, we get $10=C(2)(1)$. Thus $A=1,\ B=-5,\ C=5$, and thus

$$\int \frac{x^2+1}{(x-1)(x-2)(x-3)}dx = \ln(x-1) - 5\ln(x-2) + 5\ln(x-3) + C\ .$$

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5. Integrate $\int e^x \sin x dx$

Answer. We integrate by parts with $u=e^x,\ dv=\sin x dx$, giving us $du=e^x dx,\ v=-\cos x$, so

$$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx\ .$$

The same idea gives us

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx\ .$$

Putting this in the preceding equation gives us

$$\int e^x \sin x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx\ ,$$

from which we learn

$$\int e^x \sin x dx = \frac{1}{2}(-e^x \cos x + e^x \sin x) + C\ .$$

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6. The population of Dim Corners, Alabama has been decreasing at a rate of 4.6% per year for the past ten years. If the present population is 6,100, what was the population six years ago?

Answer. This is an exponential decay problem, with $r=.046, P(0)=6100$. We are asked to find $P(-6)$. We evaluate

$$P(-6) = 6100e^{(-.046)(-6)} = 6100e^{.0276} = 8039\ .$$

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7. Find the limit:

a) $\lim_{x\to 1} \dfrac{\cos(\pi x)+1}{(x-1)^2} =$

Answer. Since $\cos(\pi(1)) = -1$, l'Hôpital's rule applies and

$$\lim_{x\to 1} \frac{\cos(\pi x)+1}{(x-1)^2} =^{l'H} \lim_{x\to 1} \frac{-\pi\sin(\pi x)}{2(x-1)} \; .$$

Since both numerator and denominator are zero at $x = 1$, we can once again apply l'Hôpital's rule:

$$=^{l'H} \lim_{x\to 1} \frac{-\pi^2\cos(\pi x)}{2} = \frac{\pi^2}{2} \; .$$

b) $\displaystyle\int_1^\infty \frac{\ln x}{x} dx =$

Answer. Let $u = \ln x,\ du = dx/x$, so that the integral becomes $\int_0^\infty u du$. This clearly is infinite.

c) $\displaystyle\int_1^\infty \frac{dx}{x^{\frac{6}{5}}}$

Answer. $= \lim_{A\to\infty} \displaystyle\int_1^A x^{-6/5} dx = -5x^{-1/5}\big|_1^A = 5$.

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8. Find the Taylor expansion for $\int \frac{dx}{1+x^4}$ centered at $x = 0$. What is its radius of convergence?

Answer. We start with the geometric series, which has radius of convergence 1:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Now, substitute $-x^4$ for x. The radius of convergence is still 1:

$$\frac{1}{1+x^4} = \sum_{n=0}^{\infty} (-1)^n x^{4n}$$

Now, integrate both sides. The radius of convergence is still 1:

$$\int \frac{dx}{1+x^4} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{4n+1}$$

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9. Do the following series converge or diverge? Give your reasoning.

a) **Answer**. $\Sigma_{n=1}^{\infty} \dfrac{n}{n^2+1}$ diverges by comparison with the series $\sum(1/n)$:

$$\frac{n}{n^2+1} = \frac{1}{n+\frac{1}{n}} > \frac{1}{2n} \; .$$

b) **Answer**. $\Sigma_{n=1}^{\infty} \dfrac{2^n}{n!}$ converges by the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n} = \frac{2}{n+1} \to 0$$

which is less than 1.

c) **Answer**. $\Sigma_{n=1}^{\infty} \frac{n}{n^3+n^2+1}$ converges by comparison with the series $\sum(1/n^2)$:

$$\frac{n}{n^3+n^2+1} = \frac{1}{n^2+\frac{1}{n}+\frac{1}{n^2}} < \frac{1}{n^2} .$$

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10. Find the area enclosed by the curve given in polar coordinates by $r = 4\sec\theta$ from $\theta = 0$ to $\theta = \pi/3$.

Answer. Since $dA = (1/2)r^2 d\theta$:

$$Area \ = \frac{1}{2}\int_0^{\pi/3} (4\sec\theta)^2 d\theta = 8\tan\theta\Big|_0^{\pi/3} = 8\sqrt{3}$$

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11. Here is the equation of an hyperbola:

$$2x^2 - 6y^2 + 10x - 12y = 92 .$$

Find the coordinates of its center and vertices, and the slopes of its asymptotes.

Answer. First, complete the square:

$$2(x+\frac{5}{2})^2 - 6(y+1)^2 = \frac{197}{2} ,$$

so the center is at $(-5/2, -1)$, and the axis is horizontal. Dividing by 197/2 we get:

$$\frac{(x+\frac{5}{2})^2}{\frac{197}{4}} - \frac{(y+1)^2}{\frac{197}{12}} = 1 ,$$

so that

$$a = \frac{\sqrt{197}}{2},\ b = \frac{\sqrt{197}}{2\sqrt{3}}$$

and the vertices are at the points $(\frac{5}{2} \pm \frac{\sqrt{197}}{2}, -1)$ and the asymptotes have slope $\pm\frac{1}{\sqrt{3}}$.

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12. Solve the initial value problem:

$$y'' + 8y \ = \ e^{5x} \quad , \qquad y(0) = 4, y'(0) = 0 .$$

Answer. The solution of the homogeneous equation is

$$y_h = A\cos(\sqrt{8}x) + B\sin(\sqrt{8}x) .$$

To find a particular solution, try $y_p = ae^{5x}$, to find $(25a+8a)e^{5x} = e^{5x}$, so $a = 1/33$. Thus our solution is

$$y = \frac{1}{33}e^{5x} + A\cos(\sqrt{8}x) + B\sin(\sqrt{8}x) .$$

Now, the equations for the initial conditions are

$$4 = \frac{1}{33} + A \qquad 0 = \frac{5}{33} + \sqrt{8}B ,$$

giving us the solution

$$y = \frac{1}{33}e^{5x} - \frac{131}{33}\cos(\sqrt{8}x) - \frac{5}{33\sqrt{8}}\sin(\sqrt{8}x) .$$