Calculus II
Practice Exam 3, Answers

In problems 1-4, find the limits.

1. $\lim_{x\to 0} \dfrac{\cos x - 1}{x^2}$

Answer. $=^{l'H} \lim_{x\to 0} \frac{-\sin x}{2x} = -\frac{1}{2}$

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2. $\lim_{x\to \pi} \dfrac{(x-\pi)^3}{\sin x + x - \pi}$

Answer. $=^{l'H} \lim_{x\to\pi} \dfrac{3(x-\pi)^2}{\cos x + 1} =^{l'H} \lim_{x\to\pi} \dfrac{6(x-\pi)}{-\sin x} =^{l'H} \lim_{x\to\pi} \dfrac{6}{-\cos x} = 6$

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3. $\lim_{x\to\infty} x^5 e^{-x}$

Answer. $= \lim_{x\to\infty} \dfrac{x^5}{e^x} = 0,$

which converges to zero since the exponential grows faster than any polynomial.

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4. $\lim_{x\to\infty} \dfrac{\sqrt{1+x^2} - x}{x}$

Answer. $= \lim_{x\to\infty} (\sqrt{\dfrac{1}{x^2} + 1} - 1) = 0$,

since $x^{-2} \to 0$ as $x \to \infty$. We arrived at the second formulation from the first by dividing both numerator and denominator by x. Observe that, although l'Hôpital's rule applies, it doesn't get us anywhere.

In problems 5-7: Does the integral converge or diverge? If you can, find the value of the integral.

5. $\int_0^\infty xe^{-x^2} dx = \frac{1}{2}\int_0^\infty e^{-u} du = \frac{1}{2}$,

using the substitution $u = x^2,\ du = 2xdx$ and a known computation (see example 8.16).

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6. **Answer**. $\displaystyle\int_0^\infty \frac{x^2}{x^3+1} dx$ diverges , since

$$\frac{x^2}{x^3+1} = \frac{1}{x + \frac{1}{x^2}} \geq \frac{1}{2x}$$

for x sufficiently large, and our knowledge that $\int_0^\infty dx/x$ diverges.

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7. $\int_0^1 \frac{dx}{x^{9/10}}$

Answer. $= \lim_{a\to 0}\int_a^1 \frac{dx}{x^{9/10}} = \lim_{a\to 0} = \lim_{a\to 0} 10x^{1/10}\big|_a^1 = 10$

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8. Does the sequence converge or diverge?

a) $a_n = \frac{n^2}{n!}$

Answer. $a_n = \frac{n^2}{n!} = \frac{n^2}{n(n-1)(n-2)!} = (\frac{1}{1-\frac{2}{n}})\frac{1}{(n-2)!} \to 0$

since the first factor converges to 1, while the second converges to 0.

b) $b_n = \frac{\sqrt{n!}}{(n+1)^2}$

Answer. $b_n = \frac{\sqrt{n!}}{(n+1)^2} = \sqrt{\frac{n!}{(n+1)^4}} \to \infty$

because the expression under the square root sign goes to infinity (which we can show by an argument similar to that in part a).

c) $c_n = \frac{n^3-50n+1}{n^4+123n^3+1}$

Answer. $c_n = \frac{n^3-50n+1}{n^4+123n^3+1} = \frac{1-\frac{50}{n^2}+\frac{1}{n^3}}{n(1+\frac{123}{n}+\frac{1}{n^4})} \to 0$

since every factor converges to 1 except that $n \to \infty$.

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9. Does the series converge or diverge?

a) $\sum_1^\infty \frac{n^2}{n!}$

Answer. This converges by the ratio test: $\frac{(n+1)^2}{(n+1)!}\frac{n!}{n^2} = (1+\frac{1}{n})^2\frac{1}{n+1} \to 0$

which is less than 1.

b) $\sum_1^\infty \frac{\sqrt{n!}}{(n+1)^2}$

Answer. This diverges by 9b: the general term does not go to 0.

c) $\sum_{20}^\infty \frac{n^3-50n+1}{n^4+123n^3+1}$

Answer. This diverges because $\frac{n^3-50n+1}{n^4+123n^3+1} = \frac{1-\frac{50}{n^2}+\frac{1}{n^3}}{n(1+\frac{123}{n}+\frac{1}{n^4})} > \frac{1}{2n}$

eventually. By comparison with $\sum(1/n)$ the series diverges.

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10. Does the series converge or diverge?

a) $\sum_{1}^{\infty} \frac{3n+1}{n^{5/2}}$ converges
by comparison with the series $\sum(1/n^{3/2})$:

$$\frac{3n+1}{n^{5/2}} = \frac{3+\frac{1}{n}}{n^{3/2}} < \frac{4}{n^{3/2}}$$

b) $\sum_{1}^{\infty} \frac{3^n n!}{(n+1)!5^n+1}$ converges
by comparison with the geometric series:

$$\frac{3^n n!}{(n+1)!5^n+1} = \frac{1}{n+1}\left(\frac{3^n}{5^n + \frac{1}{(n+1)!}}\right) \le \left(\frac{3}{5}\right)^n$$

c) $\sum_{1}^{\infty} \frac{(2n)!(n+1)}{(2n+1)!}$ diverges
since the general term does not converge to 0:

$$\frac{(2n)!(n+1)}{(2n+1)!} = \frac{n+1}{2n+1} \to \frac{1}{2}$$

d) $\sum_{1}^{\infty} \frac{1}{n^{1/2}(3n+1)}$ converges
by comparison with the series $\sum(1/n^{3/2})$:

$$\frac{1}{n^{1/2}(3n+1)} < \frac{1}{3n^{3/2}}$$

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11. Find the radius of convergence of the series:

a) $\sum_{n=3}^{\infty} n(n-1)(n-2)x^{n-3}$
Answer. . We observe that this is the thrice differentiated geometric series, so $R = 1$. However we can use the ratio test for the coefficients:

$$\frac{(n+1)n(n-1)}{n(n-1)(n-2)} = \frac{n+1}{n-2} \to 1$$

b) $\sum_{0}^{\infty}(2^n+1)x^n$

Answer. Write down the ratio of successive coefficients and divide numerator and denominator by 2^n:

$$\frac{2^{n+1}+1}{2^n+1} = \frac{2+\frac{1}{2^n}}{1+\frac{1}{2^n}} \to 2\ ,$$

so the radius of convergence is 1/2.

c) $\sum_{1}^{\infty}\left(\frac{3n^2+1}{n^3+1}\right)(x+1)^n$
Answer. The coefficient looks like $3/n$ and so the series converges if $|x+1| < 1$, and diverges outside this interval. Thus $R = 1$.

12. Find the Maclaurin series for $(1+x)^{-3}$.

Answer. Starting with the geometric series, substitute $-x$ for x:

$$(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$$

Now, differentiate twice:

$$-(1+x)^{-2} = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$$

$$2(1+x)^{-3} = \sum_{n=2}^{\infty} (-1)^n n(n-1) x^{n-2}$$

so

$$\frac{1}{(1+x)^3} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n$$

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13. Find the Maclaurin series for $\int_0^x \arctan t dt$.

Answer. We start by substituting $-x^2$ for x in the geometric series:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Now integrate twice:

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} ,$$

$$\int_0^x \arctan t dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+2)(2n+1)} ,$$

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14. Find the Maclaurin series for $x \ln(x+1)$.

Answer. Once again start with the geometric series, with $-x$ for x

$$(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n .$$

Integrate and multiply by x:

$$\ln x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+2}}{n+1} .$$

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15. Find the terms up to fourth order for the Maclaurin series for

$$\frac{e^x}{1+x}$$

Answer. We write down the Maclaurin series for each of e^x, $1/(1+x)$, explicitly, that is, term by term, up to the fourth order:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\frac{1}{1-x} = 1 - x + x^2 - x^3 + x^4 + \cdots$$

Now, we multiply these together as if they were polynomials, relegating all terms of order greater than 4 to the $\cdots$:

$$\frac{e^x}{1+x} = (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots)(1 - x + x^2 - x^3 + x^4 + \cdots)$$

$$= (1 - x + x^2 - x^3 + x^4) + (x - x^2 + x^3 - x^4) + (\frac{x^2}{2} - \frac{x^3}{2} + \frac{x^4}{2}) + (\frac{x^3}{6} - \frac{x^4}{6}) + \frac{x^4}{24} + \cdots$$

where we have done the multiplication by successively multiplying the second series by the terms of the first. Now we collect terms;

$$\frac{e^x}{1+x} = 1 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{9}{24}x^4 + \cdots$$

(Why have all the terms in the first two parentheses , except 1,cancelled?)